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On affine motions and bar frameworks in general position

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ABSTRACT

A configuration p in r-dimensional Euclidean space is a finite collection of points $(p^1, \ldots, p^n)$ that affinely span $\mathbb{R}^r$. A bar framework, denoted by $G(p)$, in $\mathbb{R}^r$ is a simple graph G on n vertices together with a configuration p in $\mathbb{R}^r$. A given bar framework $G(p)$ is said to be universally rigid if there does not exist another configuration q in any Euclidean space, not obtained from p by a rigid motion, such that $||q^i - q^j|| = ||p^i - p^j||$ for each edge (i, j) of G.

It is known [2,7] that if configuration p is generic and bar framework $G(p)$ in $\mathbb{R}^r$ admits a positive semidefinite stress matrix S of rank $(n - r - 1)$, then $G(p)$ is universally rigid. Connelly asked [9] whether the same result holds true if the genericity assumption of p is replaced by the weaker assumption of general position. We answer this question in the affirmative in this paper.

1. Introduction

A *configuration* p in r-dimensional Euclidean space is a finite collection of points $(p^1, \ldots, p^n)$ in $\mathbb{R}^r$ that affinely span $\mathbb{R}^r$. A *bar framework* (or framework for short) in $\mathbb{R}^r$, denoted by $G(p)$, is a configuration p in $\mathbb{R}^r$ together with a simple graph G on the vertices $1, 2, \ldots, n$. For a simple graph G, we denote its node set by $V(G)$ and its edge set by $E(G)$. To avoid trivialities, we assume throughout this paper that graph G is connected and not complete.

Framework $G(q)$ in $\mathbb{R}^r$ is said to be *congruent* to framework $G(p)$ in $\mathbb{R}^r$ if configuration q is obtained from configuration p by a rigid motion. That is, if $\|q^i - q^j\| = \|p^i - p^j\|$ for all $i, j = 1, \ldots, n$, where $\|.\|$ denotes the Euclidean norm. We say that framework $G(q)$ in $\mathbb{R}^s$ is *equivalent* to framework $G(p)$

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in $\mathbb{R}^r$ if $\|q^i - q^j\| = \|p^i - p^j\|$ for all $(i,j) \in E(G)$. Furthermore, we say that framework $G(q)$ in $\mathbb{R}^r$ is *affinely-equivalent* to framework $G(p)$ in $\mathbb{R}^r$ if $G(q)$ is equivalent to $G(p)$ and configuration q is obtained from configuration p by an affine motion; i.e., $q^i = Ap^i + b$, for all $i = 1, \ldots, n$, for some $r \times r$ matrix A and an r-vector b.

A framework $G(p)$ in $\mathbb{R}^r$ is said to be *universally rigid* if there does not exist a framework $G(q)$ in any Euclidean space that is equivalent, but not congruent, to $G(p)$. The notion of a stress matrix S of a framework $G(p)$ plays a key role in the problem of universal rigidity of $G(p)$.

1.1. Stress matrices and universal rigidity

Let $G(p)$ be a framework on n vertices in $\mathbb{R}^r$. An *equilibrium stress* of $G(p)$ is a real valued function ω on $E(G)$ such that

$$\sum_{j:(i,j)\in E(G)} \omega_{ij}(p^i - p^j) = 0 \quad \text{for all } i = 1, \ldots, n. \tag{1}$$

Let ω be an equilibrium stress of $G(p)$. Then the $n \times n$ symmetric matrix $S = (s_{ij})$ where

$$s_{ij} = \begin{cases} -\omega_{ij} & \text{if } (i,j) \in E(G), \\ 0 & \text{if } i \neq j \text{ and } (i,j) \notin E(G), \\ \displaystyle\sum_{k:(i,k)\in E(G)} \omega_{ik} & \text{if } i = j, \end{cases} \tag{2}$$

is called the *stress matrix* associated with ω, or a stress matrix of $G(p)$. The following result provides a sufficient condition for the universal rigidity of a given framework.

Theorem 1.1 [6,7,1]. *Let $G(p)$ be a bar framework in $\mathbb{R}^r$, for some $r \leqslant n-2$. If the following two conditions hold:*

1. *There exists a positive semidefinite stress matrix S of $G(p)$ of rank $(n-r-1)$.*
2. *There does not exist a bar framework $G(q)$ in $\mathbb{R}^r$ that is affinely-equivalent, but not congruent, to $G(p)$.*

Then $G(p)$ is universally rigid.

Note that $(n-r-1)$ is the maximum possible value for the rank of the stress matrix S. In connection with Theorem 1.1, we mention the following result obtained in So and Ye [12] and Biswas et al [5]: Given a framework $G(p)$ in $\mathbb{R}^r$, if there does not exist a framework $G(q)$ in $\mathbb{R}^s$ ($s \neq r$) that is equivalent to $G(p)$, then $G(p)$ is universally rigid. Moreover, if $G(p)$ contains a clique of $r+1$ points in general position, then the existence of a rank-$(n-r-1)$ positive semidefinite stress matrix implies that framework $G(p)$ is universally rigid, regardless whether the other non-clique points are in general position or not.

Condition 2 of Theorem 1.1 is satisfied if configuration p is assumed to be generic (see Lemma 2.2 below). A configuration p (or a framework $G(p)$) is said to be *generic* if all the coordinates of $p^1, \ldots, p^n$ are algebraically independent over the integers. That is, if there does not exist a non-zero polynomial f with integer coefficients such that $f(p^1, \ldots, p^n) = 0$. Thus we have the following theorem.

Theorem 1.2 [7,2]. *Let $G(p)$ be a generic bar framework on n nodes in $\mathbb{R}^r$, for some $r \leqslant n-2$. If there exists a positive semidefinite stress matrix S of $G(p)$ of rank $(n-r-1)$. Then $G(p)$ is universally rigid.*

The converse of Theorem 1.2 is also true.

Theorem 1.3 [11]. *Let $G(p)$ be a generic bar framework on n nodes in $\mathbb{R}^r$, for some $r \leqslant n-2$. If $G(p)$ is universally rigid, then there exists a positive semidefinite stress matrix S of $G(p)$ of rank $(n-r-1)$.*