Linear Algebra and its Applications 438 (2013) 37–46

Contents lists available at SciVerse ScienceDirect

Linear Algebra and its Applications

journal homepage: www.elsevier.com/locate/laa

Geometric distance-regular graphs without 4-claws

S. Bang

Department of Mathematics, Yeungnam University, Gyeongsan-si, Gyeongsangbuk-do 712-749, Republic of Korea

ARTICLE INFO

Article history:
Received 9 November 2010
Accepted 12 July 2012
Available online 30 August 2012

Submitted by R.A. Brualdi

AMS classification:
05C50
05E30

Keywords:
Delsarte clique
Geometric distance-regular graph
n-Claw
Smallest eigenvalue

ABSTRACT

A non-complete distance-regular graph Γ is called geometric if there exists a set $\mathcal{C}$ of Delsarte cliques such that each edge of Γ lies in a unique clique in $\mathcal{C}$. In this paper we determine the non-complete distance-regular graphs satisfying $\max\left\{3, \frac{8}{3}(a_1+1)\right\} < k < 4a_1 + 10 - 6c_2$. To prove this result, we first show by considering non-existence of 4-claws that any non-complete distance-regular graph satisfying $\max\left\{3, \frac{8}{3}(a_1+1)\right\} < k < 4a_1+10-6c_2$ is a geometric distance-regular graph with smallest eigenvalue -3. Moreover, we classify the geometric distance-regular graphs with smallest eigenvalue -3. As an application, two feasible intersection arrays in the list of [7, Chapter 14] are ruled out.

1. Introduction

Let Γ be a distance-regular graph with valency k and let $\theta_{\min} = \theta_{\min}(\Gamma)$ be its smallest eigenvalue. Any clique C in Γ satisfies

$$|C| \leqslant 1 - \frac{k}{\theta_{\min}} \tag{1}$$

(see [7, Proposition 4.4.6 (i)]). The bound (1) is due to Delsarte, and a clique C in Γ is called a *Delsarte clique* if C contains exactly $1 - \frac{k}{\theta_{\min}}$ vertices. Godsil [11] introduced the following notion of a geometric distance-regular graph. A non-complete distance-regular graph Γ is called *geometric* if there exists a set $\mathcal{C}$ of Delsarte cliques such that each edge of Γ lies in a unique Delsarte clique in $\mathcal{C}$. In this case we say that Γ is geometric with respect to $\mathcal{C}$. There are many examples of geometric distance-regular graphs, such as bipartite distance-regular graphs, Hamming graphs, Johnson graphs, Grassmann graphs and regular near $2D$-gons.

E-mail address: sjbang@ynu.ac.kr

0024-3795/$ - see front matter
http://dx.doi.org/10.1016/j.laa.2012.07.021

In particular, the local structure of geometric distance-regular graphs plays an important role in the study of spectral characterization of some distance-regular graphs. In [1] we show that for given integer $D \geqslant 2$, any graph cospectral with the Hamming graph $H(D, q)$ is locally the disjoint union of D copies of the complete graph of size $q-1$, for q large enough. Using this result and [4], we show in [1] that the Hamming graph $H(3, q)$ with $q \geqslant 36$ is uniquely determined by its spectrum.

Neumaier [17] showed that except for a finite number of graphs, any geometric strongly regular graph with a given smallest eigenvalue $-m$, $m > 1$ integer, is either a Latin square graph or a Steiner graph (see [17] and Remark 4.4 for the definitions). An *n-claw* is an induced subgraph on $n+1$ vertices which consists of one vertex of valency n and n vertices of valency 1. Each distance-regular graph without 2-claws is a complete graph. Note that for any geometric distance-regular graph Γ with respect to $\mathcal{C}$ a set of Delsarte cliques, the number of Delsarte cliques in $\mathcal{C}$ containing a fixed vertex is $-\theta_{\min}$. Hence any geometric distance-regular graph with smallest eigenvalue -2 contains no 3-claws. Blokhuis and Brouwer [6] determined the distance-regular graphs without 3-claws. Yamazaki [21] considered distance-regular graphs which are locally a disjoint union of three cliques of size a_1+1, and for $a_1 \geqslant 1$ these graphs are geometric distance-regular graphs with smallest eigenvalue -3.

In Theorem 4.3 we determine the geometric distance-regular graphs with smallest eigenvalue -3. We now state the main result of this paper.

Theorem 1.1. *Let Γ be a non-complete distance-regular graph with valency k, diameter D and intersection numbers a_i, b_i, c_i $(1 \leqslant i \leqslant D)$. If Γ satisfies*

$$\max\left\{3, \frac{8}{3}(a_1+1)\right\} < k < 4a_1 + 10 - 6c_2$$

then Γ is one of the following, where $\iota(\Gamma)$ and h are as defined in (2) and (3).

(i) *A Steiner graph $S_3(\alpha-3)$, i.e., a geometric strongly regular graph with parameters $\left(\frac{(2\alpha-3)(\alpha-2)}{3}, 3\alpha-9, \alpha, 9\right)$, where $\alpha \geqslant 36$ and $\alpha \not\equiv 1 \pmod 3$.*
(ii) *A Latin square graph $LS_3(\alpha)$, i.e., a geometric strongly regular graph with parameters $(\alpha^2, 3(\alpha-1), \alpha, 6)$, where $\alpha \geqslant 24$.*
(iii) *The generalized hexagon of order $(8,2)$ with $\iota(\Gamma) = \{24, 16, 16; 1, 1, 3\}$.*
(iv) *One of the two generalized hexagons of order $(2,2)$ with $\iota(\Gamma) = \{6, 4, 4; 1, 1, 3\}$.*
(v) *A generalized octagon of order $(4,2)$ with $\iota(\Gamma) = \{12, 8, 8, 8; 1, 1, 1, 3\}$.*
(vi) *The Johnson graph $J(\alpha, 3)$, where $\alpha \geqslant 20$.*
(vii) *$\iota(\Gamma) = \{3\alpha+3, 2\alpha+2, \alpha+2-\beta; 1, 2, 3\beta\}$, where $\alpha \geqslant 6$ and $\alpha \geqslant \beta \geqslant 1$.*
(viii) *The halved Foster graph with $\iota(\Gamma) = \{6, 4, 2, 1; 1, 1, 4, 6\}$.*
(ix) *$D = h+2 \geqslant 4$ and*

$$(c_i, a_i, b_i) = \begin{cases} (1, \alpha, 2\alpha+2) & \text{for } 1 \leqslant i \leqslant h \\ (2, 2\alpha+\beta-1, \alpha-\beta+2) & \text{for } i = h+1 \\ (3\beta, 3\alpha-3\beta+3, 0) & \text{for } i = h+2 \end{cases}, \text{ where } \alpha \geqslant \beta \geqslant 2.$$

(x) *$D = h+2 \geqslant 3$ and*

$$(c_i, a_i, b_i) = \begin{cases} (1, \alpha, 2\alpha+2) & \text{for } 1 \leqslant i \leqslant h \\ (1, \alpha+2\beta-2, 2\alpha-2\beta+4) & \text{for } i = h+1 \\ (3\beta, 3\alpha-3\beta+3, 0) & \text{for } i = h+2 \end{cases}, \text{ where } \alpha \geqslant \beta \geqslant 2.$$