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More on G-matrices

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ABSTRACT

The first author and Hall defined recently a G-matrix as a real nonsingular matrix A such that there exist diagonal matrices D_1 and D_2 for which $(A^T)^{-1} = D_1AD_2$. The class of G-matrices was shown to possess interesting properties. In this paper, some new characterizations are found and extensions to rectangular and complex matrices are discussed.

1. Introduction

The first author and Hall defined in [2] the class of so called G-matrices. There, a real nonsingular matrix A was called G-matrix if there exist diagonal matrices D_1 and D_2 such that $(A^T)^{-1} = D_1AD_2$.

It was observed in [2] that the following holds:

Theorem 1.1. *All orthogonal matrices are G-matrices.*

Theorem 1.2. *All nonsingular diagonal matrices are G-matrices.*

Theorem 1.3. *If A is a G-matrix, then both A^T and A^{-1} are G-matrices.*

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Theorem 1.4. *If A is an $n \times n$ G-matrix and D is an $n \times n$ nonsingular diagonal matrix, then both AD and DA are G-matrices.*

Theorem 1.5. *If A is an $n \times n$ G-matrix and P is an $n \times n$ permutation matrix, then both AP and PA are G-matrices.*

Theorem 1.6. *If A is a G-matrix, then both matrices A and A^{-T} have the same zero, non-zero structure. Thus the zero, non-zero structures of A and A^{-1} are symmetric to each other.*

Theorem 1.7. *The direct sum of G-matrices is again a G-matrix.*

The product form of the definition leads to the following two theorems.

Theorem 1.8. *Compound matrices of a G-matrix are also G-matrices.*

Theorem 1.9. *The Kronecker product of G-matrices is a G-matrix.*

Also, the following theorem was proved.

Theorem 1.10. *A 2×2 matrix is G-matrix if and only if it is nonsingular and has four or two nonzero entries.*

As is well known, Cauchy matrices are matrices of the form $C = [c_{ij}]$, where $c_{ij} = \frac{1}{x_i+y_j}$ for some numbers x_i and y_j. We shall restrict ourselves to square, say $n \times n$, Cauchy matrices. Of course, such matrices are defined only if $x_i + y_j \neq 0$ for all pairs of indices i, j, and it is well known that C is nonsingular if and only if all the numbers x_i are mutually distinct and all the numbers y_j are mutually distinct. In [2], the following theorem was proved.

Theorem 1.11. *Every nonsingular Cauchy matrix C for which both vectors $C^{-1}e$ and $C^{-T}e$ have no coordinate zero is a G-matrix; here, e is the vector of all ones.*

Let us recall [1, Observation 1] that the assumptions about the vectors $C^{-1}e$ and $C^{-T}e$ are superfluous. Indeed, it is well known that if $C = \frac{1}{x_i+y_j}$, then the inverse $C^{-1} = [\gamma_{ij}]$, where

$$\gamma_{ij} = (x_j + y_i)\frac{\prod_{\ell\neq i}(x_j + y_\ell)\prod_{k\neq j}(y_i + x_k)}{\prod_{\ell\neq j}(x_j - x_\ell)\prod_{k\neq i}(y_i - y_k)}. \tag{1}$$

The right-hand side of (1) can be written as $\frac{1}{x_j+y_i}U_jV_i$, where

$$U_j = (x_j + y_j)\prod_{k\neq j}\frac{x_j + y_k}{x_j - x_k}, \tag{2}$$

$$V_i = (x_i + y_i)\prod_{k\neq i}\frac{y_i + x_k}{y_i - y_k}. \tag{3}$$

If we thus define the diagonal matrices D_1, D_2 as $D_1 = \text{diag}(U_1, \ldots, U_n)$, $D_2 = \text{diag}(V_1, \ldots, V_n)$, then indeed $(C^T)^{-1} = D_1CD_2$. Thus C is a G-matrix.

2. Results

Let us start with the following observation.