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Positive and negative inertia index of a graph ☆

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ABSTRACT

In this paper, the positive and negative inertia index of trees, unicyclic graphs and bicyclic graphs are discussed, the methods of calculating them are obtained, and an inequality about the difference between positive and negative inertia index is proved. Moreover, a conjecture is proposed.

1. Introduction

Throughout the paper, graphs are simple, i.e., without loops and multiple edges. Let $G = (V, E)$ be a simple graph of order n with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and edge set $E = E(G)$. The adjacency matrix $A = (a_{ij})_{n\times n}$ of G is defined as follows: $a_{ij} = 1$ if v_i is adjacent to v_j, and $a_{ij} = 0$ otherwise. The eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of A are said to be the eigenvalues of the graph G, and to form the spectrum of this graph. The number of positive, negative and zero eigenvalues in the spectrum of the graph G are called positive inertia index, negative inertia index and nullity of the graph G, and are denoted by $p(G)$, $n(G)$ and $\eta(G)$, respectively. Obviously $p(G) + n(G) + \eta(G) = n$.

There have been diverse studies on the nullity of a graph [1–3,5–8,10–14]; it is related to the stability of molecular represented by the graph. However, there is very little literature on positive and negative inertia index of a graph. As we know, if G is a bipartite graph, then $p(G) = n(G) = \frac{1}{2}(n - \eta(G))$. Otherwise, $p(G)$, $n(G)$ and $\eta(G)$ do not have this relationship. In this paper, the positive and negative inertia index of trees, unicyclic graphs and bicyclic graphs are discussed, methods of calculating them

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are obtained, and an inequality about the difference between positive and negative inertia index is proved. Moreover, a conjecture is proposed.

We first introduce some concepts and notations. Let $G = (V, E)$ be a simple graph. Given a subset $W \subseteq V(G)$, the subgraph (of G) induced by W, written as $G[W]$, is defined to be the graph with vertex set W and edge set $\{xy \in E(G) : x \in W \text{and } y \in W\}$. We write $G - U$ for the graph obtained from G by removing the vertices of U and all edges incident to any of them. Sometimes we use the notation $G - G_1$ instead of $G - V(G_1)$, where G_1 is a subgraph of G. We define the union of two disjoint graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, denoted by $G_1 \cup G_2$, as the graph with vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2$. A tree (unicyclic graph and bicyclic graph) is a simple connected graph in which the number of edges equals the number of vertices plus -1 (0 and 1, respectively). A complete graph, a path, a cycle and a star of order n are denoted by K_n, P_n, C_n and $K_{1,n-1}$, respectively. An isolated vertex is sometime denoted by K_1.

This paper is organized as follows: in Section 2, some necessary lemmas are given, in Section 3, calculating the positive and negative inertia index of trees and unicyclic graphs is discussed, in Section 4, calculating the positive and negative inertia index of bicyclic graphs is discussed, in Section 5, an inequality about the difference between positive and negative inertia index is obtained, and a conjecture is proposed.

2. Some lemmas

In this section, we cite some previous results. Suppose A, B are two real symmetric matrices of order n, if there exists an invertible matrix P of order n such that $P'AP = B$, then we say that A is congruent to B, denoted by $A \simeq B$.

Lemma 2.1 [9]. *Let A, B be two real symmetric matrices of order n, such that $A \simeq B$. Then $p(A) = p(B)$, $n(A) = n(B)$, $\eta(A) = \eta(B)$.*

The following lemma is well known.

Lemma 2.2. *Let $G = G_1 \cup G_2 \cup \ldots \cup G_t$, where $G_i (i = 1, 2, \ldots, t)$ are connected components of G. Then $p(G) = \sum_{i=1}^{t} p(G_i)$, $n(G) = \sum_{i=1}^{t} n(G_i)$, $\eta(G) = \sum_{i=1}^{t} \eta(G_i)$.*

Lemma 2.3. *Let G be a graph containing a pendant vertex, and let H be the induced subgraph of G obtained by deleting the pendant vertex together with the vertex adjacent to it. Then $p(G) = p(H) + 1$, $n(G) = n(H) + 1$, $\eta(G) = \eta(H)$.*

Proof. Suppose G is a graph containing a pendant vertex. Then the adjacency matrix $A(G)$ has the following form

$$A(G) = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 1 & 0 & x_1 & \ldots & x_{n-2} \\ 0 & x_1 & & & \\ \vdots & \vdots & & A(H) & \\ 0 & x_{n-2} & & & \end{bmatrix},$$

where $x_i \in \{0, 1\}$, $i = 1, 2, \ldots, n-2$. It is easy to show that $A(G)$ is congruent to

$$A(P_2 \cup H) = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & & & \\ \vdots & \vdots & & A(H) & \\ 0 & 0 & & & \end{bmatrix}.$$