Linear Algebra and its Applications 438 (2013) 436–442

Contents lists available at SciVerse ScienceDirect

Linear Algebra and its Applications

journal homepage: www.elsevier.com/locate/laa

# Inequalities for absolute value operators

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ARTICLE INFO

*Article history:*
Received 5 May 2012
Accepted 10 August 2012
Available online 27 September 2012

Submitted by C.-K. Li

*AMS classification:*
47A63

*Keywords:*
Dunkl–Williams inequality
Operator inequalities
Operator absolute value

ABSTRACT

We present refinements of an inequality which is due to Saito and Tominaga [Linear Algebra Appl. 432 (2010) 3258–3264], and other inequalities for absolute value operators.


## 1. Introduction

In this note we mainly adopt the notation and terminology in [1,2]. For convenience, recall that, as usual, let $B(H)$ be the algebra of all bounded linear operators on a complex separable Hilbert space $H$. For $A \in B(H)$, we denote by $|A|$ the absolute value operator of $A$, that is, $|A| = (A^*A)^{\frac{1}{2}}$, where $A^*$ is the adjoint operator of $A$. A self-adjoint operator $A \in B(H)$ is said to be positive if $(Ax, x) \geq 0$ for all $x \in H$. We write $A \geq 0$ if $A$ is positive. We denote by $[AH]$ the closure of $AH$ and by $P_{[AH]}$ the orthogonal projection onto $[AH]$. Let $A = U|A|$ be the polar decomposition of $A \in B(H)$ with $U^*U = P_{[|A|H]}$. Throughout this note, we assume that $p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Let $A, B \in B(H)$ with polar decomposition $A = U|A|$ and $B = V|B|$. By using operator Bohr inequality [3, Corollary 1], Saito and Tominaga [1, Theorem 2.3] obtained an inequality for absolute value operators as follows:

$$|(U - V)|A||^2 \leq p|A - B|^2 + q(|A| - |B|)^2. \tag{1.1}$$

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http://dx.doi.org/10.1016/j.laa.2012.08.025

It is a generalization of the following inequality:

$$|A|A|^{-1} - B|B|^{-1}|^2 \le |A|^{-1}(p|A-B|^2 + q(|A|-|B|)^2)|A|^{-1}, \tag{1.2}$$

which is due to Pečarić and Rajić [2, Theorem 2.1]. As a consequence of the inequality (1.2), Pečarić and Rajić [2, Corollary 2.3] gave a Dunkl–Williams type inequality for absolute value operators:

$$|A|A|^{-1} - B|B|^{-1}| \le (|A|^{-1}(2|A-B|^2 + 2(|A|-|B|)^2)|A|^{-1})^{\frac{1}{2}}.$$

For more information on the well-known Dunkl–Williams inequality [4] and operator versions of the Dunkl–Williams inequality, the reader is referred to [1,2,5–7] and the references therein.

In this note, we give refinements of the inequality (1.1) and lower estimates for $|(U-V)|A||^2$.

## 2. Main results

We begin this section with the following result.

**Theorem 2.1.** *Let $A, B \in B(H)$ with polar decomposition $A = U|A|$ and $B = V|B|$. If $p, q > 1$, then*

$$|(U-V)|A||^2 \le |A-B|^2 + (|A|-|B|)^2 - (T+T^*) \le p|A-B|^2 + q(|A|-|B|)^2, \tag{2.1}$$

*where*

$$T = (|A|-|B|)V^*(A-B).$$

**Proof.** Since $V^*V \le I$, we have

$$\begin{aligned} |(U-V)|A||^2 &= |A-B-V(|A|-|B|)|^2 \\ &= |A-B|^2 + |V(|A|-|B|)|^2 - (T+T^*) \\ &= |A-B|^2 + (|A|-|B|)V^*V(|A|-|B|) - (T+T^*) \\ &\le |A-B|^2 + (|A|-|B|)^2 - (T+T^*). \end{aligned}$$

This proves the first part of (2.1).

Next, we prove the second part of (2.1). By a small calculation we know that

$$\frac{q}{p} = q-1, \qquad \frac{p}{q} = p-1,$$

and so

$$\begin{aligned} p|A-B|^2 &+ q(|A|-|B|)^2 - (|A-B|^2 + (|A|-|B|)^2 - (T+T^*)) \\ &= (p-1)|A-B|^2 + (q-1)(|A|-|B|)^2 + (T+T^*) \\ &\ge (p-1)|A-B|^2 + (q-1)|V(|A|-|B|)|^2 + (T+T^*) \\ &= \left|\sqrt{p-1}(A-B) + \sqrt{q-1}V(|A|-|B|)\right|^2 \\ &\ge 0. \end{aligned}$$

This completes the proof. □

**Remark 2.1.** Obviously, the inequality (2.1) is a refinement of the inequality (1.1).

**Remark 2.2.** Interchanging the operators $A$ and $B$ in the inequality (2.1), we have

$$|(U-V)|B||^2 \le |A-B|^2 + (|A|-|B|)^2 - (T+T^*) \le p|A-B|^2 + q(|A|-|B|)^2,$$