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Characterization of Lie derivations on von Neumann algebras ☆

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ABSTRACT

Let $\mathcal{M}$ be any von Neumann algebra without central summands of type I_1. For any scalar ξ, a characterization of any additive map L on $\mathcal{M}$ satisfies $L(AB-\xi BA)=L(A)B-\xi BL(A)+AL(B)-\xi L(B)A$ whenever $AB=0$ is given: there exists an additive derivation φ such that, (1) if $\xi=1$, then $L=\varphi+f$, where f is an additive map into the center vanishing on $[A,B]$ with $AB=0$; (2) if $\xi=0$, then $L(I)\in\mathcal{Z}(\mathcal{M})$ and $L(A)=\varphi(A)+L(I)A$ for all A; (3) if ξ is rational and $\xi\neq 0,1$, then $L=\varphi$; (4) if ξ is not rational, then $\varphi(\xi I)=\xi L(I)$ and $L(A)=\varphi(A)+L(I)A$.

1. Introduction

Let $\mathcal{R}$ be an associative ring (or an algebra over a field $\mathbb{F}$). Recall that an additive (a linear) map δ from $\mathcal{R}$ into itself is called an additive (a linear) derivation if $\delta(AB)=\delta(A)B+A\delta(B)$ for all A, $B\in\mathcal{R}$. More generally, an additive (a linear) map L from $\mathcal{R}$ into itself is called an additive (a linear) Jordan derivation if $L(AB+BA)=L(A)B+AL(B)+L(B)A+BL(A)$ for all $A,B\in\mathcal{R}$ (equivalently, $L(A^2)=L(A)A+AL(A)$ for all $A\in\mathcal{R}$ if the characteristic of $\mathcal{R}$ is not 2, i.e., $\mathcal{R}$ is 2-torsion free); is called a Lie derivation if $L([A,B])=[L(A),B]+[A,L(B)]$ for all $A,B\in\mathcal{R}$, where $[A,B]=AB-BA$ is the Lie product of A and B. The problem of how to characterize the linear (additive) Jordan (Lie) derivations of rings and algebras has received many mathematicians' attention for many years. Brešar in [2] proved

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that every additive Lie derivation on a prime ring $\mathcal{R}$ with characteristic not 2 can be decomposed as $\tau + \zeta$, where τ is a derivation from $\mathcal{R}$ into its central closure and ζ is an additive map of $\mathcal{R}$ into the extended centroid $\mathcal{C}$ sending commutators to zero. Johnson [9] proved that every continuous linear Lie derivation from a C^*-algebra $\mathcal{A}$ into a Banach $\mathcal{A}$-bimodule M is standard, that is, can be decomposed as the form $\tau + h$, where $\tau : \mathcal{A} \to M$ is a derivation and h is a linear map from $\mathcal{A}$ into the center of M vanishing at each commutator. Mathieu and Villena [15] showed that every linear Lie derivation on a C^*-algebra is standard. In [18] Qi and Hou proved that the same is true for additive Lie derivations of nest algebras on Banach spaces. For other results, see [3,7] and the references therein.

Recently, there have been a number of papers on the study of conditions under which derivations of rings or operator algebras can be completely determined by the action on some elements concerning products (see [4,8,11,13,17] and the references therein). For Lie derivations, some works were also done. A linear (an additive) map $L : \mathcal{R} \to \mathcal{R}$ is said to be Lie derivable at a point Z if $L([A, B]) = [L(A), B] + [A, L(B)]$ for any $A, B \in \mathcal{R}$ with $[A, B] = Z$. Clearly, this definition is not valid for some Z, for instance, for $Z = I$, as the unit I may not be a commutator $[A, B]$ in general. It is also obvious that the condition of maps Lie derivable at some point is much weaker than the condition of being a Lie derivation. Qi and Hou [19] discussed such linear maps on $\mathcal{J}$-subspace lattice algebras. Lu and Jing in [14] gave another kind of characterization for Lie derivations as follows. Let X be a Banach space with $\dim X \geqslant 3$ and $\mathcal{B}(X)$ the algebra of all bounded linear operators acting on X. It is proved in [14] that if $\delta : \mathcal{B}(X) \to \mathcal{B}(X)$ is a linear map satisfying $\delta([A, B]) = [\delta(A), B] + [A, \delta(B)]$ for any $A, B \in \mathcal{B}(X)$ with $AB = 0$ (resp. $AB = P$, where P is a fixed nontrivial idempotent), then $\delta = d + \tau$, where d is a derivation of $\mathcal{B}(X)$ and $\tau : \mathcal{B}(X) \to \mathbb{C}I$ is a linear map vanishing at commutators $[A, B]$ with $AB = 0$ (resp. $AB = P$). Later, this result was generalized to the maps on triangular algebras and prime rings in [10] and [20] respectively. Since factor von Neumann algebras are prime, as a consequence of the result for prime rings, all additive maps δ on factor von Neumann algebras satisfying $\delta([A, B]) = [\delta(A), B] + [A, \delta(B)]$ for any A, B with $AB = 0$ are characterized. However the proof for factor von Neumann algebras is not valid anymore for general von Neumann algebras. So, it is natural to ask what happens when the concerned von Neumann algebras are not factor.

Let $\mathcal{A}$ be an algebra over a field $\mathbb{F}$. For a scalar $\xi \in \mathbb{F}$ and for $A, B \in \mathcal{A}$, if $AB = \xi BA$, we say that A commutes with B up to a factor ξ. The notion of commutativity up to a factor for pairs of operators is an important concept and has been studied in the context of operator algebras and quantum groups (Refs. [5,12]). Motivated by this, a binary operation $[A, B]_\xi = AB - \xi BA$, called ξ-Lie product of A and B, was introduced in [18]. An additive (a linear) map $L : \mathcal{A} \to \mathcal{A}$ is called an additive (a linear) ξ-Lie derivation if $L([A, B]_\xi) = [L(A), B]_\xi + [A, L(B)]_\xi$ for all $A, B \in \mathcal{A}$. This conception unifies several well known notions. It is clear that a ξ-Lie derivation is a derivation if $\xi = 0$; is a Lie derivation if $\xi = 1$; is a Jordan derivation if $\xi = -1$. The structure of ξ-Lie derivations was characterized in triangular algebras and prime algebras in [18] and [21] respectively. Particularly, we got a characterization of ξ-Lie derivations on Banach space nest algebras and standard operator algebras.

Thus, more generally, one may ask what is the structure of additive (linear) maps L that satisfy $L([A, B]_\xi) = [L(A), B]_\xi + [A, L(B)]_\xi$ for any A, B with $AB = 0$? The purpose of the present paper is to study this question for maps on von Neumann algebras and characterize all such maps on general von Neumann algebras. Note that every map on a commutative von Neumann algebra is a Lie derivation. So it is reasonable to confine our attention to the von Neumann algebras that have no central summands of type I_1.

This paper is organized as follows. Let $\mathcal{M}$ be a von Neumann algebra without central summands of type I_1. In Section 2, we deal with the case $\xi = 1$, that is, the case of Lie product, and show that every additive map $L : \mathcal{M} \to \mathcal{M}$ satisfies $L([A, B]) = [L(A), B] + [A, L(B)]$ for any $A, B \in \mathcal{M}$ with $AB = 0$ if and only if it has the form $L = \varphi + f$, where $\varphi : \mathcal{M} \to \mathcal{M}$ is an additive derivation and $f : \mathcal{M} \to \mathcal{Z}(\mathcal{M})$, the center of $\mathcal{M}$, is an additive map vanishing on each commutator $[A, B]$ whenever $AB = 0$ (Theorem 2.1). Section 3 is devoted to discussing the case of $\xi \neq 1$. We show that every additive map $L : \mathcal{M} \to \mathcal{M}$ satisfies that $L([A, B]_\xi) = [L(A), B]_\xi + [A, L(B)]_\xi$ for any $A, B \in \mathcal{M}$ with $AB = 0$ if and only if $L(I) \in \mathcal{Z}(\mathcal{M})$ and, (1) $\xi = 0$, there exists an additive derivation φ such that $L(A) = \varphi(A) + L(I)A$ for all $A \in \mathcal{M}$; (2) $\xi \in \mathbb{C}$ is rational and $\xi \neq 0, 1$, L is an additive derivation; (3) $\xi \in \mathbb{C}$ is not rational, there exists an additive derivation φ satisfying $\varphi(\xi I) = \xi L(I)$ such that