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Functional identities of degree 2 in triangular rings

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ABSTRACT

Let R be a triangular ring. The problem of describing the form of additive maps $F_1, F_2, G_1, G_2 : R \to R$ satisfying functional identity $F_1(x)y+F_2(y)x+xG_2(y)+yG_1(x) = 0$ for all $x, y \in R$ is considered. As an application generalized inner biderivations and commuting additive maps of certain triangular rings are determined.

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1. Introduction

The theory of functional identities was initiated by Brešar in the beginning of 90s and it was further developed by Beidar, Brešar, Chebotar, and Martindale. For a full account on the theory of functional identities the reader is referred to the book [6]. Let R be a nonempty subset of a unital ring Q with center $Z(Q)$. Let $F_1, F_2, G_1, G_2 : R \to Q$ be maps such that

$$F_1(x)y + F_2(y)x + xG_2(y) + yG_1(x) = 0 \tag{1.1}$$

for all $x, y \in R$. This is a basic functional identity, which was one of the first functional identities studied (in prime rings). The usual goal in the theory of functional identities is to describe set-theoretic maps satisfying certain identities. Thus, our maps F_1, F_2, G_1, G_2 are considered as unknowns. It is easy to see that maps of the form

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$$\begin{aligned} F_1(x) &= xq_1 + \alpha_1(x) \\ F_2(x) &= xq_2 + \alpha_2(x) \\ G_1(x) &= -q_2x - \alpha_1(x) \\ G_2(x) &= -q_1x - \alpha_2(x), \end{aligned} \tag{1.2}$$

where $q_1, q_2 \in Q$, $\alpha_1, \alpha_2 : R \to Z(Q)$, present a solution of (1.1). Following [6] we say that each solution of the form (1.2) is a *standard solution* of (1.1). In [5] Brešar considered functional identity (1.1) in the context of prime rings. Suppose that I is an ideal of a prime ring R with extended centroid C. Under the assumption that $F_1, F_2, G_1, G_2 : I \to R$ are additive modulo C he proved that (1.1) has only standard solution (1.2). Using this result he described the form of generalized inner biderivations of prime rings [5, Theorem 4.7]. Somewhat later, Zhang et al. [9] considered functional identity (1.1) in nest algebras. They discovered that in a certain class of nest algebras functional identity (1.1) has only the standard solution. Consequently, they described the form of generalized inner biderivations for these nest algebras. On the other hand, each additive map which is *commuting*, i.e. $F(x)x - xF(x) = 0$ for all $x \in R$, satisfies

$$F(x)y + F(y)x - xF(y) + yF(x) = 0$$

for all $x, y \in R$, which is just a special example of (1.1). We say that each commuting map of the form $F(x) = \lambda x + \mu(x)$, where λ is a central element in Q and $\mu : R \to Z(Q)$, is of the *standard form*. In [4] Brešar proved that each commuting additive map on a prime ring has standard form. The same conclusion was obtained for semiprime rings by Ara and Mathieu [1]. In 2001 Cheung [7] considered commuting linear maps on triangular algebras. He determined the class of triangular algebras for which every commuting linear map has standard form.

Motivated by these results we consider functional identity (1.1) in a triangular ring R. The main purpose of this paper is to determine the form of maps $F_1, F_2, G_1, G_2 : R \to R$ satisfying (1.1). Our main result, Theorem 2.2, is applied to the problem of describing commuting maps of triangular rings. Consequently, we extend the result of Cheung [7, Theorem 2] and also [3, Remark 2.8]. Our main theorem also enables us to determine the form of generalized inner biderivations for a certain class of triangular rings. In particular, we extend the result of Zhang et al. [9, Theorem 3.1] in the context of nest algebras.

2. The main result

A unital ring R with a nontrivial idempotent e is a *triangular ring*, if eRf is a faithful (eRe, fRf)-bimodule and $fRe = 0$, where f denotes $1 - e$. Each triangular ring R has the following Peirce decomposition:

$$R = eRe \oplus eRf \oplus fRf.$$

Note that a ring R is triangular if and only if there exist unital rings A, B and a unital faithful (A, B)-bimodule M such that R is isomorphic to the ring

$$\mathrm{Tri}(A, M, B) := \left\{ \begin{pmatrix} a & m \\ 0 & b \end{pmatrix}; a \in A, m \in M, b \in B \right\}$$

with the usual matrix addition and multiplication.

By [7, Proposition 3] the center of a triangular ring R equals

$$Z(R) = \{c \in eRe + fRf \mid c(exf) = (exf)c \text{ for all } x \in R\}.$$

Moreover, $Z(R)e \subseteq Z(eRe)$ and $Z(R)f \subseteq Z(fRf)$ and there exists a unique ring isomorphism $\tau : Z(R)e \to Z(R)f$ such that $ece(exf) = (exf)\tau(ece)$ for all $x \in R$. In order to prove our main theorem we need the following result.

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