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On the traces of elements of modular group

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ABSTRACT

We prove a conjecture by Bergweiler and Eremenko on the traces of elements of modular group in this paper.

1. Introduction

Bergweiler and Eremenko made a remarkable conjecture on the traces of elements of modular group in [1]. The main result of this paper is to prove their conjecture. We expect this result to have future applications in some fields such as control theory.

Let $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$. These two matrices generate the free group which is called $\Gamma(2)$, the principal congruence subgroup of level 2. With arbitrary integers $m_j \neq 0$, $n_j \neq 0$, consider the trace of the product

$$p_k(m_1, n_1, \dots, m_k, n_k) = tr(A^{m_1} B^{n_1} \cdots A^{m_k} B^{n_k}).$$

It is easy to see that p_k is a polynomial in $2k$ variables with integer coefficients. This polynomial can be written explicitly though the formula is somewhat complicated.

Choosing an arbitrary sequence σ of $2k$ signs $\pm$, we make a substitution

$$p_k^\sigma(x_1, y_1, \dots, x_k, y_k) = p_k(\pm(1 + x_1), \pm(1 + y_1), \dots, \pm(1 + x_k), \pm(1 + y_k)).$$

Our main theorem is the following one. Which was conjectured by Bergweiler and Eremenko [1].

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Theorem 1.1. *The polynomial p_k, for every $k > 0$, has the property that for every σ, all the coefficients of the polynomial p_k^σ are of the same sign, that is, the sequence of coefficients of p_k^σ has no sign changes.*

We prove the theorem by induction on k. However it is not easy to pass from "level k" to "level $k+1$" since that p_k has the above property does not simply imply that p_{k+1} has the same one. The idea here is to substitute p_k's with a suitable set of polynomials containing the p_k's so that the difficulty disappears. This idea is explained in Section 2 (see Proposition 2.2) and the theorem is showed in Section 3.

2. Traces

2.1. BE polynomials

Set

$$F_k = \begin{pmatrix} f_k & h_k \\ t_k & g_k \end{pmatrix} = A^{x_1}B^{y_1}A^{x_2}B^{y_2}\cdots A^{x_k}B^{y_k}$$

where $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$. Then the trace $p_k = trF_k = f_k + g_k$ and all f_k, h_k, t_k, g_k are the polynomials in $2k$ variables $x_1, y_1, \ldots, x_k, y_k$ with integer coefficients whose explicit formula can be found in [1].

A sequence σ of $2k$ signs $\pm$ can be viewed as a function $\sigma : \{1, 2, \ldots, 2k\} \to \{1, -1\}$. For any polynomial f in variables $x_1, y_1, \ldots, x_k, y_k$, set

$$f^\sigma = f(\sigma(1)(1+x_1), \sigma(2)(1+y_1), \ldots, \sigma(2k-1)(1+x_k), \sigma(2k)(1+y_k))$$

Definition 2.1. A polynomial f in $2k$ variables is said to be a BE polynomial if for arbitrary sequence σ of $2k$ signs, all the coefficients of f^σ have the same sign.

Let $Mat(2, 2)$ be the set of 2×2 matrices over $\mathbf{R}$, the set of real numbers. Denote by F_k^σ the matrix $\begin{pmatrix} f_k^\sigma & h_k^\sigma \\ t_k^\sigma & g_k^\sigma \end{pmatrix}$. If $M = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \in Mat(2, 2)$, then

$$\begin{aligned} tr(F_kM) &= af_k + bh_k + ct_k + dg_k \\ tr(F_k^\sigma M) &= af_k^\sigma + bh_k^\sigma + ct_k^\sigma + dg_k^\sigma \end{aligned}$$

Write

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad A_2 = \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix} \quad A_3 = \begin{pmatrix} 2 & -1 \\ 0 & 0 \end{pmatrix},$$

$$A_4 = \begin{pmatrix} 3 & 2 \\ -2 & -1 \end{pmatrix} \quad A_5 = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} \quad A_6 = \begin{pmatrix} 5 & -2 \\ -2 & 1 \end{pmatrix}.$$

Note that

$$A_4 + A_5 = 4A_2,\ A_4 + A_6 = 4A_3^t,\ A_4^t + A_5 = 4A_2^t,\ A_4^t + A_6 = 4A_3, A_2 + A_3 = 4A_1, \tag{2.1}$$

$$A_4 = -A^{-1}B^{-1},\ A_4^t = -AB,\ A_5 = AB^{-1},\ A_6 = A^{-1}B. \tag{2.2}$$

Let S be a subset of $Mat(2, 2)$, we have