Linear Algebra and its Applications 436 (2012) 1154–1176

Contents lists available at SciVerse ScienceDirect

Linear Algebra and its Applications

journal homepage: www.elsevier.com/locate/laa

On the sandpile group of the cone of a graph ☆

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ARTICLE INFO

Article history:
Received 16 March 2011
Accepted 12 July 2011
Available online 22 October 2011

Submitted by R.A. Brualdi

AMS classification:
Primary: 05C25
Secondary: 05C50, 05E99

Keywords:
Sandpile group
Critical group
Graph homomorphism
Cartesian product
Hypercube

ABSTRACT

In this article, we study the sandpile group of the cone of a graph. After introducing the concept of uniform homomorphism of graphs we prove that every surjective uniform homomorphism of graphs induces an injective homomorphism between their sandpile groups. Also, we establish a relationship between the sandpile group of the cone of the cartesian product of graphs and the sandpile group of the cone of their factors. As an application of these results we obtain an explicit description of a set of generators of the sandpile group of the cone of the hypercube.

1. Introduction

The *sandpile models* were firstly introduced by Bak et al. [3,4], and have been studied under several names in statistical physics, theoretical computer science, algebraic graph theory, and combinatorics.

The *abelian sandpile model* of a graph was introduced by Dhar [19], which generalizes the sandpile model of a grid given in [3]. The abelian sandpile model of Dhar [19] begins with a connected graph $G = (V, E)$ and a distinguished vertex $s \in V$, called the sink. Dhar [19] showed that the set of some configurations (a configurations of G is a vector in $\mathbb{N}^{V \setminus s}$), called *recurrent configurations*, with the vertex-by-vertex sum as a binary operation forms a finite abelian group, called the sandpile group of G. It follows from Kirchhoff's Matrix-Tree theorem (see e.g. [7]) that the order of the sandpile group of

☆ The first author was partially supported by CONACyT and the second author was partially supported by SNI.
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0024-3795/$ - see front matter
doi:10.1016/j.laa.2011.07.030

a graph G is the number of spanning trees of G. Mainly, the abelian sandpile group has been studied under the name of sandpile group, denoted by $SP(G, s)$, and critical group, denoted by $K(G)$. It has been also studied under other names, such as Jacobian group, Picard group, dollar game, see, for instance [8,9,28,29].

The sandpile group has been completely determined for some family of graphs, see, for instance [28, 32,29,25,9,33,35,26,14]. The sandpile group of the cartesian product has received special interest, for instance the following cartesian products of graphs it has been determined: $P_4 \times C_n$ [13], $K_3 \times C_n$ [24], $K_m \times P_n$ [27], $C_4 \times C_n$ [38], and $K_m \times C_n$ [17,39]. The abstract structure of the sandpile group has been partially described for the hypercube [2] and the cartesian product of complete graphs [25]. In [16] it was proved that the sandpile group of a dual graph G^* is isomorphic to the sandpile group of G. Also, in [6] there are established some relations between the sandpile group of a graph G and the sandpile group of its line graph. In particular, they proved that if G is non-bipartite and regular, then $K(\mathbf{line}(G))$ is completely determined as a function of $K(G)$. Finally, in [30] a relationship between the eigenvalues and eigenvectors of the Laplacian matrix of a graph and their sandpile group is established.

Given a natural number n, the *n-cone* of a graph G, denoted by $c_n(G)$, is the graph obtained from G when we add a new vertex s to G and n parallel edges between the new vertex s and all the vertices of G. If $n = 1$ we simply write $c(G)$ instead of $c_1(G)$. In this article, we study the sandpile group of the cone of a graph. In particular, we give a partial description of the sandpile group of the cone of the cartesian product of graphs as a function of the sandpile group of the cone of their factors. Also, we introduce the concept of uniform homomorphism of graphs and prove that every surjective uniform homomorphism of graphs induces an injective homomorphism between their sandpile groups. As an application of these two results we obtain an explicit description of a set of generators of the sandpile group of the cone of the hypercube of dimension d.

A *graph* G is a pair (V, E), where V is a finite set and E is a subset of the set of unordered pair of elements of V. The elements of V and E are called *vertices* and *edges*, respectively. If $e = \{x, y\}$, then x and y are *incident* to e, x and y are the *ends* of e and x and y are *adjacents*. The *multiplicity* between two vertices u and v of a graph, denoted by $m_{u,v}$, is the number of edges with ends u and v. The *degree* of a vertex $x \in G$, denoted by $d_G(x) = d(x)$, is the number of incident edges to x.

A graph $G' = (V', E')$ is a *subgraph* of the graph $G = (V, E)$, if $V' \subseteq V$ and $E' \subseteq E$. An *induced* subgraph $G[V'] = (V', E')$ is a subgraph of $G = (V, E)$ such that every edge $e \in E$ that has its ends in V' is in E'.

The article is organized as follows. In Section 2, the concepts of graph theory that will be needed in the rest of the article are introduced. We also give the combinatorial and algebraic definitions of the sandpile group of G with sink s_G.

In Section 3, we introduce the concept of uniform homomorphism of graphs. Let G and H be two graphs and $V \subseteq V(H)$. A V-uniform homomorphism between G and H, is a mapping $f : V(G) \to V(H)$ such that for all $x \in V$ and $y \in V(H)$

$$d_{G[\{u\} \cup S_y]}(u) = m_{x,y} \quad \text{for all} \quad u \in S_x = f^{-1}(x)$$

and $f : V(G) \setminus f^{-1}(V) \to V(H) \setminus V$ is the identity isomorphism. After introducing the concept of a V-uniform homomorphism, we prove the main theorem of this section.

Theorem 3.5. *If $f : G \to H$ is a surjective V-uniform homomorphism with $f^{-1}(s_H) = \{s_G\}$ and $s_H \notin V \subset V(H)$ such that $V(H) \setminus V$ is a stable set, then the induced mapping $\widetilde{f} : SP(H, s_H) \to SP(G, s_G)$, given by*

$$\widetilde{f}(\mathbf{c})_v = \begin{cases} \mathbf{c}_{f(v)} & \text{if } f(v) \in V, \\ \deg(f) \cdot c_{f(v)} & \text{if } f(v) \notin V, \end{cases}$$

is an injective homomorphism of groups.