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The classification of Leonard triples of QRacah type

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ABSTRACT

Let $\mathbb{K}$ denote an algebraically closed field. Let V denote a vector space over $\mathbb{K}$ with finite positive dimension. By a Leonard triple on V we mean an ordered triple of linear transformations in $\mathrm{End}(V)$ such that for each of these transformations there exists a basis of V with respect to which the matrix representing that transformation is diagonal and the matrices representing the other two transformations are irreducible tridiagonal. There is a family of Leonard triples said to have QRacah type. This is the most general type of Leonard triple. We classify the Leonard triples of QRacah type up to isomorphism. We show that any Leonard triple of QRacah type satisfies the $\mathbb{Z}_3$-symmetric Askey–Wilson relations.

1. Leonard pairs and Leonard systems

We begin by recalling the notion of a Leonard pair. We will use the following terms. Let X denote a square matrix. Then X is called *tridiagonal* whenever each nonzero entry lies on either the diagonal, the subdiagonal, or the superdiagonal. Assume X is tridiagonal. Then X is called *irreducible* whenever each entry on the subdiagonal is nonzero and each entry on the superdiagonal is nonzero.

We now define a Leonard pair. For the rest of this paper $\mathbb{K}$ will denote an algebraically closed field.

Definition 1.1 [4, Definition 1.1]**.** Let V denote a vector space over $\mathbb{K}$ with finite positive dimension. By a *Leonard pair* on V, we mean an ordered pair of linear transformations $A : V \to V$ and $A^* : V \to V$ that satisfy both (i), (ii) below.

(i) There exists a basis for V with respect to which the matrix representing A is irreducible tridiagonal and the matrix representing A^* is diagonal.
(ii) There exists a basis for V with respect to which the matrix representing A^* is irreducible tridiagonal and the matrix representing A is diagonal.

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Note 1.2. According to a common notational convention A^* denotes the conjugate-transpose of A. We are not using this convention. In a Leonard pair (A, A^*) the linear transformations A and A^* are arbitrary subject to (i), (ii) above.

For the rest of this paper we fix an integer $d \geqslant 0$. Let $\mathrm{Mat}_{d+1}(\mathbb{K})$ denote the $\mathbb{K}$-algebra consisting of all $d+1$ by $d+1$ matrices that have entries in $\mathbb{K}$. We index the rows and columns by $0, 1, \ldots, d$. We let $\mathbb{K}^{d+1}$ denote the $\mathbb{K}$-vector space consisting of all $d+1$ by 1 matrices that have entries in $\mathbb{K}$. We index the rows by $0, 1, \ldots, d$. We view $\mathbb{K}^{d+1}$ as a left module for $\mathrm{Mat}_{d+1}(\mathbb{K})$. For the rest of the paper let V denote a vector space over $\mathbb{K}$ that has dimension $d+1$. Let $\mathrm{End}(V)$ denote the $\mathbb{K}$-algebra consisting of all linear transformations from V to V. Let $\{v_i\}_{i=0}^d$ denote a basis for V. For $X \in \mathrm{End}(V)$ and $Y \in \mathrm{Mat}_{d+1}(\mathbb{K})$, we say Y *represents* X *with respect to* $\{v_i\}_{i=0}^d$ whenever $Xv_j = \sum_{i=0}^d Y_{ij} v_i$ for $0 \leqslant j \leqslant d$. For $A \in \mathrm{End}(V)$, by an *eigenvalue* of A we mean a root of the characteristic polynomial of A. We say that A is *multiplicity-free* whenever it has $d+1$ distinct eigenvalues. Assume A is multiplicity-free. Let $\{\theta_i\}_{i=0}^d$ denote an ordering of the eigenvalues of A. For $0 \leqslant i \leqslant d$ let V_i denote the eigenspace of A associated with θ_i. Define $E_i \in \mathrm{End}(V)$ such that $(E_i - I)V_i = 0$ and $E_i V_j = 0$ for $j \neq i$ $(0 \leqslant j \leqslant d)$. Here I denotes the identity of $\mathrm{End}(V)$. We call E_i the *primitive idempotent* of A associated with θ_i.

Lemma 1.3 [4, Lemma 1.3]. *Let* (A, A^*) *denote a Leonard pair on* V. *Then each of* A, A^* *is multiplicity-free.*

We now define a Leonard system.

Definition 1.4 [4, Definition 1.4]. By a *Leonard system* on V we mean a sequence $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ that satisfies (i)–(v) below.

(i) Each of A, A^* is a multiplicity-free element in $\mathrm{End}(V)$.
(ii) $\{E_i\}_{i=0}^d$ is an ordering of the primitive idempotents of A.
(iii) $\{E_i^*\}_{i=0}^d$ is an ordering of the primitive idempotents of A^*.
(iv) $$E_i A^* E_j = \begin{cases} 0 & \text{if } |i-j| > 1, \\ \neq 0 & \text{if } |i-j| = 1 \end{cases} \qquad (0 \leqslant i, j \leqslant d).$$
(v) $$E_i^* A E_j^* = \begin{cases} 0 & \text{if } |i-j| > 1, \\ \neq 0 & \text{if } |i-j| = 1 \end{cases} \qquad (0 \leqslant i, j \leqslant d).$$

We refer to d as the *diameter* of Φ and say Φ is *over* $\mathbb{K}$.

Definition 1.5 [4, Definition 1.8]. Let $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a Leonard system on V. For $0 \leqslant i \leqslant d$ let θ_i (resp. θ_i^*) denote the eigenvalue of A (resp. A^*) associated with E_i (resp. E_i^*). We call $\{\theta_i\}_{i=0}^d$ (resp. $\{\theta_i^*\}_{i=0}^d$) the *eigenvalue sequence* (resp. *dual eigenvalue sequence*) of Φ.

Definition 1.6 [4, Definition 2.5]. Let $(A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a Leonard system. Define

$$a_i = \mathrm{tr}(AE_i^*), \qquad a_i^* = \mathrm{tr}(A^*E_i) \qquad (0 \leqslant i \leqslant d),$$

where tr denotes trace.

The scalars $\{a_i\}_{i=0}^d$, $\{a_i^*\}_{i=0}^d$ have the following interpretation.

Lemma 1.7 [5, Lemma 10.2]. *With reference to Definition 1.6,*

$$E_i^* A E_i^* = a_i E_i^* \qquad (0 \leqslant i \leqslant d),$$
$$E_i A^* E_i = a_i^* E_i \qquad (0 \leqslant i \leqslant d).$$