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Reverse order laws in C*-algebras☆

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ABSTRACT

In this paper, we offer purely algebraic necessary and sufficient conditions for reverse order laws for generalized inverses in C*-algebras, extending rank conditions for matrices and range conditions for Hilbert space operators.

1. Introduction

Let $\mathcal{A}$ be a complex unital C*-algebra. An element $a \in \mathcal{A}$ is said to be *regular* (in the sense of von Neumann) if there exists $b \in \mathcal{A}$ for which $aba = a$; any such b is called an *inner inverse* of a.

An element $x \in \mathcal{A}$ which satisfies the four Penrose equations [11],

$$(1)\ axa = a, \quad (2)\ xax = x, \quad (3)\ (ax)^* = ax, \quad (4)\ (xa)^* = xa,$$

if it exists, is called the Moore–Penrose inverse of a and is denoted by $a^\dagger$. From the definition of the Moore–Penrose inverse, we conclude that both $a^\dagger a$ and $aa^\dagger$ are *projections*, where by a projection we mean an element $p \in \mathcal{A}$ which is a hermitian idempotent, i.e., such that $p^2 = p = p^*$. A Moore–Penrose inverse is unique if it exists, and this is the case exactly when $a \in \mathcal{A}$ is regular (see [7]):

$$a \text{ is regular} \iff a\mathcal{A} \text{ is closed} \iff a^\dagger \text{ exists}.$$

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If $a \in \mathcal{A}$ is regular and $c \in \mathcal{A}$ is arbitrary, then $aa^\dagger\mathcal{A}$ is the range of the projection $aa^\dagger$, so that

$$aa^\dagger c = c \Leftrightarrow c\mathcal{A} \subset a\mathcal{A}.$$

For $K \subseteq \{1, 2, 3, 4\}$, we shall call $x \in \mathcal{A}$ a K-*inverse* of $a \in \mathcal{A}$ if it satisfies the Penrose equation (j) for each $j \in K$. We shall write aK for the collection of all K-inverses of $a \in \mathcal{A}$, and a^K for an unspecified element $x \in aK$. Also, by $bK \cdot aK$ we denote the setwise product

$$bK \cdot aK = \{xy : x \in bK,\ y \in aK\}.$$

If p and q are projections in $\mathcal{A}$ then we can represent elements $x \in \mathcal{A}$ as 2×2 matrices over $\mathcal{A}$, writing

$$x = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}_{p,q},$$

where $x_1 = pxq, x_2 = px(1-q), x_3 = (1-p)xq, x_4 = (1-p)x(1-q)$; note that $x = x_1+x_2+x_3+x_4$.

The reverse order law for the Moore–Penrose inverse seems first to have been studied by Greville [5] , in the '60s , giving a necessary and sufficient condition for the reverse order law

$$(AB)^\dagger = B^\dagger A^\dagger, \tag{1.1}$$

for matrices A and B. This has been followed (see [4]) by further equivalents of (1.1). Sun and Wei [7] extended these investigations to the case of weighted Moore–Penrose inverses of matrices, and Hartwig [8] and Tian [13,14] to the case of products of three and more matrices, respectively. Koliha et al. [10] studied the reverse order law for the product of Moore–Penrose invertible elements in the setting of rings with involution. The next step was to extend the discussion of (1.1) to the more general case of reverse order law for K-inverses where $K \subseteq \{1, 2, 3, 4\}$. Werner [15] studied necessary and sufficient conditions for the case $K = \{1\}$ in the setting of matrices. The cases $K = \{1, 3\}$ and $K = \{1, 4\}$ were considered by Wei and Guo [17] who obtained some results using product singular value decomposition (P-SVD) of matrices, and later, also in the settings of matrices, by Takane et al. [12] who discovered some new necessary and sufficient conditions using other techniques. Djordjević [3] considered the cases $K = \{1, 3\}$ and $K = \{1, 4\}$ but for bounded operators on Hilbert spaces. Wang [16] studied mixed-type reverse order law for matrices when $K = \{1, 3\}$. Xiong and Zheng [18] considered the cases $K = \{1, 2, 3\}$ and $K = \{1, 2, 4\}$ for products of two matrices and their techniques involved expressions for maximal and minimal ranks of the generalized Schur complement.

In this paper, for the first time, we extend the discussion of the reverse order law for K-inverses in the cases $K \in \{\{1, 3\}, \{1, 4\}, \{1, 2, 3\}, \{1, 2, 4\}\}$ to C*-algebra elements. The main motivation for the choice of C*-algebras as the setting for the research is to generalize all the previous results concerning this subject (for matrices or bounded operators on Hilbert spaces) using some new techniques without any rank or range conditions. Our necessary and sufficient conditions are purely algebraic, but reduce to the rank conditions of Xiong and Zheng [18] in the matrix algebra case, and to the range conditions of Djordjević [3] in the case of Hilbert space operators. As a byproduct we will see that the $\{1, 2, 3\}$ conditions imply the $\{1, 3\}$ conditions, and the $\{1, 2, 4\}$ conditions the $\{1, 4\}$ conditions.

2. Characterizations

We begin by extending to C*-algebras characterizations of sets aK, given in [1] for complex matrices and in [3] for Hilbert space operators:

Lemma 2.1. *Let $a \in \mathcal{A}$ be regular and $b \in \mathcal{A}$. Then $b \in a\{1, 3\}$ if and only if $a^\dagger ab = a^\dagger$.*

Proof. If $a^\dagger ab = a^\dagger$, then $aba = aa^\dagger a = a$ and $(ab)^* = (aa^\dagger)^* = aa^\dagger = ab$. Conversely, we have that $ab = b^*a^* = b^*(aa^\dagger a)^* = b^*a^*aa^\dagger = abaa^\dagger = aa^\dagger$, so $a^\dagger ab = a^\dagger$. □

Lemma 2.1 can be expressed by saying

$$a\{1, 3\} = \{a^\dagger + (1 - a^\dagger a)y : y \in \mathcal{A}\}. \tag{2.1}$$