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Detection of boundary curves on the piecewise smooth boundary surface of three dimensional solids

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ABSTRACT

Suppose that Ω is a three-dimensional solid with boundary surface $S = S_1 \cup \cdots \cup S_q$, where each S_r is a smooth surface with boundary curve Γ_r. Multiscale directional representation systems (e.g., shearlets) are able to capture the essential geometry of Ω by precisely identifying the boundary set

$$\mathcal{N} = \{(p, n_r(p)) : p \in S_r, r = 1, \ldots, q\},$$

where $n_r(p)$ denotes the normal vector to the surface S_r at p. This property has resulted in the successful application of multiscale directional methods in a variety of image processing problems, since edges and boundary sets are usually the most informative features in many types of multidimensional data. However, existing methods are ill-suited to capture those edge-type singularities in the three-dimensional setting resulting from the intersection of piecewise smooth boundary surfaces. In this paper, we introduce a new multiscale directional system based on a modification of the shearlet framework and prove that the associated continuous transform has the ability to precisely identify both the location and orientation of the boundary curves Γ_r from the solid Ω. This paper extends a number of results appeared in the literature in recent years to the challenging problem of extracting curvilinear singularities in three-dimensional objects and is motivated by image analysis problems arising from areas including biomedical and seismic imaging and astronomy.

1. Introduction

Objects with discontinuities along curvilinear edges and surface boundaries appear in a variety of imaging applications. For example, in biomedical imaging, the objects of interest are cells, tissues and other organs; in this case, changes in molecular structures identifying each object are represented as edges and surfaces. In seismic imaging, the objects of interest are the material properties of the Earth's subsurface as a function

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of depth and these properties change discontinuously across a system of layer boundaries. In astronomical images, the objects of interest include intricate patterns with filaments, clusters, and sheet-like arrangements of galaxies encompassing large nearly empty regions. Notice that, in all such applications, the discontinuities occurring along edges and surfaces are the most informative features and, in many cases, the only structures one is really interested in recovering from data.

Over the past decade, a number of "directional multiscale systems" were introduced to provide improved framework for the representation of multivariate functions containing edge-type discontinuities. The *ridgelets* [2] and *beamlets* [6], for example, were introduced to represent more efficiently lines crossing an image. Other prominent constructions are the *curvelets* [3] and *shearlets* [24,14] that provide (near) optimally sparse approximations for images with curvilinear edges by combining multiscale analysis and high directional sensitivity. Due to their ability to sparsely represent curvilinear edges, methods based on these representations are particularly useful for the study of edge-dominated phenomena and often outperform more traditional multiscale methods in many image processing applications (cf. [8,9]).

Perhaps the true potential of such directional multiscale systems is best illustrated when the associated continuous transforms are applied to the analysis of singularities. The continuous curvelet transform, in particular, resolves the *wavefront set* of a distribution in two dimensions [4]. The continuous shearlet transform, in addition to satisfying the latter property [22], has the ability to precisely identify the set of discontinuities of a large class of multivariate functions. More precisely, let $f = \chi_\Omega$, where Ω is a bounded region in $\mathbb{R}^2$ or $\mathbb{R}^3$ with a piece-wise smooth boundary $S = \partial\Omega$. Then the continuous shearlet transform of f identifies both the location and orientation of the boundary set S by its asymptotic decay at fine scales [15–17,19]. These theoretical results have lead to a number of successful applications in problems of edge detection and feature extraction [5,23,26,30]. Note however that the ability to detect the set of singularities of functions and distributions is useful beyond these applications. Consider, for example, the problem of "geometric separation" which aims to break up complex data into geometrically distinct components. It was recently shown that the solution to this problem relies on the ability to detect and separate different types of singularities, e.g., pointwise singularities vs. curvilinear ones [7,21]. These observations are the foundation for several remarkable applications to image inpainting and morphological component analysis [13,20,28,27].

Motivated by the same types of applied problems, in this paper, we examine the more challenging problem of extracting curvilinear singularities in 3-dimensional objects, which is not covered by existing results. To be more precise about our setting, suppose that Ω is a three dimensional solid with boundary surface $S = S_1 \cup \cdots \cup S_q$, where each S_r is a smooth surface with boundary curve Γ_r. Let $n_r(p)$ denote the normal vector to S_r at p and $t_r(p)$ denote the tangent vector to Γ_r at p and write

$$\mathcal{N} = \big\{ \big(p, n_r(p)\big) : p \in S \big\}$$

and

$$\mathcal{T} = \big\{ \big(p, t_r(p)\big) : p \in \Gamma_r, r = 1, \ldots, q \big\}.$$

The goal of this paper is to extract the collection $\mathcal{T}$ from the solid region Ω.

On the surface, our setting is similar to reference [29] that deals with the application of directional multiscale transforms to astronomical data restoration. In this reference, the authors heuristically introduce a variant of the curvelet transform for handling singularities forming one-dimensional structures in $\mathbb{R}^3$ and apply this system to problems of denoising and inpainting of astronomical data. Note that, while several numerical illustrations are presented in [29], their approach is purely heuristic. By contrast, in this paper we develop a rigorous theoretical framework for the detection of singularities forming one-dimensional structures in $\mathbb{R}^3$ which are subsets of singularities forming 2-dimensional structures in $\mathbb{R}^3$.

It turns out that, while existing directional multiscale methods do an excellent job of detecting the boundary set $\mathcal{N}$ from Ω, they cannot detect $\mathcal{T}$ since they are designed to deal with different types of