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Spike detection from inaccurate samplings

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ABSTRACT

This article investigates the support detection problem using the LASSO estimator in the space of measures. More precisely, we study the recovery of a discrete measure (spike train) from few noisy observations (Fourier samples, moments, etc.) using an ℓ_1-regularization procedure. In particular, we provide an explicit quantitative localization of the spikes.

1. Introduction

1.1. Super-resolution

Imaging experiments can be subject to device limitations where one cannot observe enough information in order to recover fine details. For instance, in optical imaging, the physical limitations are evaluated by the resolution. This latter measures the minimal distance between lines that can be distinguished. Hence, the details below the resolution limit seem unreachable. The super-resolution phenomenon is the ability to recover the information beyond the physical limitations. Surprisingly, if the object of interest is simple, e.g. a discrete measure, then it is possible to override the resolution limit. In particular, the reader may think of important questions in applied harmonic analysis such as the problem of source separation. Many companion applications in astronomy, medical imaging and single molecule imaging in 3D microscopy are at

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stake, see [16,15,21] and references therein. Hence, theoretical guarantees of source detection are of crucial importance in practice.

In this paper, we prove quantitative detection guarantees from noisy observations (Fourier samples, moments samples). Furthermore, these quantitative estimates can be computed using a tractable algorithm, called BLASSO.

1.2. Previous works

The theoretical analysis of the ℓ_1-regularization in the space of measures was initiated by Donoho [8]. Few years after, rates of convergence in super-resolution have been investigated by P. Doukhan, E. Gassiat and one author of this present paper [9,12]. They considered the exact reconstruction of a nonnegative measure and derived results when one only knows the values of a finite number of linear functionals at the target measure. Moreover, they study stability with respect to a metric for weak convergence. Likewise, two authors of this paper [7] proved that k spikes trains can be faithfully resolved from $m = 2k + 1$ samples (Fourier samples, Stieltjes transformation, Laplace transform, etc.) by using an ℓ_1-minimization method.

Following recent proposal [3,4,17,19] on inverse problems regularization in Banach spaces, we consider convergence rates in Bregman divergence. On a more general note, inverse problems on the space of measures are now well understood, see [13,20] for instance. We capitalize on these earlier works to construct our analysis. In particular, we use them to give quantitative localizations of the recovered spikes, which is new.

In the super-resolution frame, the important paper [6] shows that if the spikes are well “separated” then there exists a dual certificate, i.e. an ℓ_∞-constrained trigonometric polynomial that interpolates the phase of the weights at the spikes locations. This construction provides a quadratic isolation condition, see Definition 2.2, of the spikes which is crucial for ℓ_1-minimization in the space of measures. In a predating paper [7], the authors investigates ℓ_1-minimization with different types of measurements: trigonometric, polynomial, Laplace transform, etc. In view of application, the recent works [5,22] derive results in ℓ_1 and ℓ_2 prediction, i.e. the estimation of the input frequencies. Moreover, note that noise robustness of support recovery is proved in [10].

A postdating paper [11] bounds the support detection error for a constrained formulation of the ℓ_1-minimization in the space of measures as in Theorem 2.2. Regarding unconstrained formulation, a second postdating paper [22] studies spikes detection for the Fourier sampling under a Gaussian noise model. The authors provide an optimal rate for the ℓ_2-prediction, namely the ℓ_2 distance between the recovered Fourier coefficients and the original ones.

The aforementioned results suggest that the recovered spike locations should be close to the input support. This is investigated, for the first time, in this paper for an unconstrained ℓ_1-minimization problem under a general sampling scheme.

1.3. General model and notation

Let $(\mathbb{T}, d)$ be a compact metric space homeomorphic to either the interval $([0, 1], |\,.\,|)$ or the unit circle $\mathbb{S}^1$, which is identified to the metric space $(\mathbb{R} \bmod(1), d(\cdot,\cdot))$ via the mapping $z = e^{\mathrm{i}2\pi t}$. In this latter case, the distance d is taken around the circle. Let Δ be a complex measure on $\mathbb{T}$ with discrete support of size s. In particular, the measure Δ has polar decomposition, see [18] for a definition:

$$\Delta = \sum_{k=1}^{s} \Delta_k \exp(\mathrm{i}\theta_k)\delta_{T_k}, \tag{1.1}$$