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Hamiltonian deformations of Gabor frames: First steps

Maurice A. de Gosson

University of Vienna, Faculty of Mathematics, NuHAG, Austria

A R T I C L E I N F O

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A B S T R A C T

Gabor frames can advantageously be redefined using the Heisenberg–Weyl operators familiar from harmonic analysis and quantum mechanics. Not only does this redefinition allow us to recover in a very simple way known results of symplectic covariance, but it immediately leads to the consideration of a general deformation scheme by Hamiltonian isotopies (*i.e.* arbitrary paths of non-linear symplectic mappings passing through the identity). We will study in some detail an associated weak notion of Hamiltonian deformation of Gabor frames, using ideas from semiclassical physics involving coherent states and Gaussian approximations. We will thereafter discuss possible applications and extensions of our method, which can be viewed – as the title suggests – as the very first steps towards a general deformation theory for Gabor frames.

1. Introduction

The theory of Gabor frames (or Weyl–Heisenberg frames as they are also called) is a rich and expanding topic of applied harmonic analysis. It has numerous applications in time–frequency analysis, signal theory, and mathematical physics. The aim of this article is to initiate a systematic study of the symplectic transformation properties of Gabor frames, both in the linear and nonlinear cases. Strangely enough, the use of symplectic techniques in the theory of Gabor frames is often ignored; one example (among many others) being Casazza's seminal paper [7] on modern tools for Weyl–Heisenberg frame theory, where the word "symplectic" does not appear a single time in the 127 pages of this paper! This is of course very unfortunate: it is a thumb-rule in mathematics and physics that when symmetries are present in a theory their use always leads to new insights in the mechanisms underlying that theory. To name just one single example, the study of fractional Fourier transforms belongs to the area of symplectic and metaplectic analysis and geometry (see Section 3.4); remarking this would avoid to many authors unnecessary efforts and complicated calculations. On the positive side, there are however (a few) exceptions to this refusal to include symplectic techniques in applied harmonic analysis: for instance, in Gröchenig's treatise [26] the metaplectic representation is used to

E-mail addresses: maurice.de.gosson@univie.ac.at, maurice.degosson@gmail.com.

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study various symmetries in time frequency analysis, and the recent paper by Pfander et al. [50] elaborates on earlier work [29] by Han and Wang, where symplectic transformations are exploited to study various properties of Gabor frames.

In this paper we consider deformations of Gabor systems using *Hamiltonian isotopies*. A Hamiltonian isotopy is a curve $(f_t)_{0\leqslant t\leqslant 1}$ of diffeomorphisms of phase space $\mathbb{R}^{2n}$ starting at the identity, and such that there exists a (usually time-dependent) Hamiltonian function H such that the (generalized) phase flow $(f_t^H)_t$ determined by the Hamilton equations

$$\dot{x} = \partial_p H(x,p,t), \qquad \dot{p} = -\partial_x H(x,p,t) \tag{1}$$

consists of the mappings f_t for $0 \leqslant t \leqslant 1$. In particular Hamiltonian isotopies consist of symplectomorphisms (or canonical transformations, as they are called in physics). Given a Gabor system $\mathcal{G}(\phi,\Lambda)$ with window (or atom) ϕ and lattice Λ we want to find a working definition of the deformation of $\mathcal{G}(\phi,\Lambda)$ by a Hamiltonian isotopy $(f_t)_{0\leqslant t\leqslant 1}$. While it is clear that the deformed lattice should be the image $\Lambda_t = f_t(\Lambda)$ of the original lattice Λ, it is less clear what the deformation $\phi_t = f_t(\phi)$ of the window ϕ should be. A clue is however given by the linear case: assume that the mappings f_t are linear, *i.e.* symplectic matrices S_t; assume in addition that there exists an infinitesimal symplectic transformation X such that $S_t = e^{tX}$ for $0 \leqslant t \leqslant 1$. Then $(S_t)_t$ is the flow determined by the Hamiltonian function

$$H(x,p) = -\frac{1}{2}(x,p)^T JX(x,p) \tag{2}$$

where J is the standard symplectic matrix. It is well-known that in this case there exists a one-parameter group of unitary operators $(\widehat{S}_t)_t$ satisfying the operator Schrödinger equation

$$i\hbar\frac{\mathrm{d}}{\mathrm{d}t}\widehat{S}_t = H(x,-i\hbar\partial_x)\widehat{S}_t$$

where the formally self-adjoint operator $H(x,-i\hbar\partial_x)$ is obtained by replacing formally p with $-i\hbar\partial_x$ in (2); the matrices S_t and the operators $\widehat{S}_t$ correspond to each other via the metaplectic representation of the symplectic group. This suggests that we define the deformation of the initial window ϕ by $\phi_t = \widehat{S}_t\phi$. It turns out that this definition is satisfactory, because it allows to recover, setting $t = 1$, known results on the image of Gabor frames by linear symplectic transformations. This example is thus a good guideline; however one encounters difficulties as soon as one want to extend it to more general situations. While it is "reasonably" easy to see what one should do when the Hamiltonian isotopy consists of an arbitrary path of symplectic matrices (this will be done in Section 4), it is not clear at all what a "good" definition should be in the general nonlinear case: this is discussed in Section 4.3, where we suggest that a natural choice would be to extend the linear case by requiring that ϕ_t should be the solution of the Schrödinger equation

$$i\hbar\frac{\mathrm{d}}{\mathrm{d}t}\phi_t = \widehat{H}\phi_t$$

associated with the Hamiltonian function H determined by the equality $(f_t)_{0\leqslant t\leqslant 1} = (f_t^H)_{0\leqslant t\leqslant 1}$; the Hamiltonian operator $\widehat{H}$ would then be associated with the function H by using, for instance, the Weyl correspondence. Since the method seems to be difficult to study theoretically and to implement numerically, we propose what we call a notion of *weak deformation*, where the exact definition of the transformation $\phi \longmapsto \phi_t$ of the window ϕ is replaced with a correspondence used in semiclassical mechanics, and which consists in propagating the "center" of a sufficiently sharply peaked initial window ϕ (for instance a coherent state, or a more general Gaussian) along the Hamiltonian trajectory. This definition coincides with the definition already given in the linear case, and has the advantage of being easily computable using the