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The stochastic properties of ℓ^1-regularized spherical Gaussian fields

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ABSTRACT

Convex regularization techniques are now widespread tools for solving inverse problems in a variety of different frameworks. In some cases, the functions to be reconstructed are naturally viewed as realizations from random processes; an important question is thus whether such regularization techniques preserve the properties of the underlying probability measures. We focus here on a case which has produced a very lively debate in the cosmological literature, namely Gaussian and isotropic spherical random fields, and we prove that neither Gaussianity nor isotropy are conserved in general under convex regularization based on ℓ^1 minimization over a Fourier dictionary, such as the orthonormal system of spherical harmonics.

1. Introduction

Let $T : M \to \mathbb{R}$ be a square integrable function on a manifold M, and assume that the following is observed:

$$T^{obs} := \mathcal{A}T + n, \tag{1}$$

where $\mathcal{A} : L^2(M) \to L^2(M)$ is a linear operator that can represent, for instance, a blurring convolution or a mask setting some values of the function T to zero, while $n : M \to \mathbb{R}$ denotes observational noise. Recovering T from observations on T^{obs} is a standard example of a linear inverse problem, and it is now very common to pursue a solution for (1) by means of convex/ℓ^1-regularization procedures. More precisely, we can proceed by postulating that the signal T can be sparsely represented in a given dictionary Ψ, e.g.,

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$T = \Psi\alpha_0$ where the vector α_0 is assumed to be sparse in a suitable sense, and then solving the ℓ^1-regularized problem

$$\alpha^{reg} := \arg\min_{\alpha}\left\{\lambda\|\alpha\|_{\ell^1} + \frac{1}{2}\left\|T^{obs} - \mathcal{A}\Psi\alpha\right\|^2_{L^2(S^2)}\right\}, \tag{2}$$

which can be viewed for instance as a form of Basis Pursuit Denoising [9] or a variation of the Lasso algorithm introduced in the statistical literature by [32]. Often the following alternative formulation is considered:

$$\alpha^{reg} := \arg\min_{\alpha}\left\{\|\alpha\|_{\ell^1}\right\} \quad \text{subject to} \quad \left\|T^{obs} - \mathcal{A}\Psi\alpha\right\|_{L^2(S^2)} \leq \varepsilon, \tag{3}$$

for some $\varepsilon > 0$; it is known that there exists a bijection $\lambda \leftrightarrow \varepsilon$ such that (2) and (3) have the same solution [29]; as argued below, in the framework of the present paper this equivalence is understood to hold with very high probability, in a sense to be made rigorous later. Many authors have worked on related regularization problems over the last two decades – a very incomplete list includes [23,11,15,17,22,33], see for instance [29], Chapter 7 for more references and a global overview. These results are also connected to the rapidly growing literature on compressive sensing, see, e.g., [13,6,8,24,25].

In many applied fields, it is customary to view T as the realization of a random field, and the reconstruction problems (2) and (3) are usually just the first steps before statistical data analysis (e.g., estimation and testing) is implemented. In other words, T is viewed as a random object on a probability space $(\Omega, \Im, P)$, $T(\omega, x) := T : \Omega \times M \to \mathbb{R}$; hence it becomes important to verify that $T^{reg} := \Psi\alpha^{reg}$, $T^{reg} : \Omega \times M \to \mathbb{R}$, is close to T in a meaningful probabilistic sense. For instance, let M be a homogeneous space of a compact group $\mathcal{G}$; a natural question is the following:

Problem 1. Assume that the field T is Gaussian and isotropic, e.g., the probability laws of $T(.)$ and $T^g(.) = T(g.)$ are the same for all $g \in \mathcal{G}$. Is the random field T^{reg} Gaussian and isotropic?

The scenario we have described fits very well, for instance, the current situation in the cosmological literature, in particular in the field of Cosmic Microwave Background (CMB) data analysis. The latter can be viewed as a snapshot picture of the Universe at the so-called age of recombination, e.g. 3.7×10^5 years after the Big Bang (some 13 billion years ago); its observation has been made possible by satellite experiments such as WMAP [7] and Planck [26], which have raised an enormous amount of theoretical and applied interest. CMB is usually viewed as a single realization of a Gaussian isotropic random field on the sphere, e.g., $M = S^2$ and $\mathcal{G} = SO(3)$, the group of rotations in $\mathbb{R}^3$; this assumption is very well motivated by our physical understanding of the early Universe, i.e. the theory of inflation (see for instance [12]). Observations are corrupted by observational noise and various forms of convolutions (e.g., instrumental beams, masked regions) and a number of efforts have been devoted to solving (1) under these circumstances. In this setting, algorithms such as (2) and (3) have been widely proposed, in some cases (see e.g., [1,14,29,30] and the references therein) taking as a dictionary the orthonormal system of spherical harmonics $\{Y_{\ell m}\}$. As well-known, the latter are eigenfunctions of the spherical Laplacian $\Delta_{S^2} Y_{\ell m} = -\ell(\ell+1)Y_{\ell m}$ and lead to the spectral representation

$$T(x) = \sum_{\ell=0}^{\infty} T_\ell(x) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} a_{\ell m} Y_{\ell m}(x).$$

Under Gaussianity and isotropy, this representation holds in the mean square sense and the random coefficients are Gaussian and independent with variance $E a_{\ell m}\bar{a}_{\ell' m'} = C_\ell \delta_\ell^{\ell'}\delta_m^{m'}$, the sequence $\{C_\ell\}$ representing the angular power spectrum (see for instance [21]). For finite variance fields we have $E\{T^2\} = \sum_\ell \frac{2\ell+1}{4\pi} C_\ell < \infty$; for definiteness and notational convenience, in what follows we shall assume that for