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On noncompact quasi Yamabe gradient solitons

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ABSTRACT

We study τ-quasi Yamabe gradient solitons on complete noncompact Riemannian manifolds. We prove several scalar curvature estimates under some conditions and get a nonlocal collapsing result based on the gradient estimate of the potential function. We also derive a decay theorem and a finite topological type result.

1. Introduction

Soliton metrics are special solutions of geometric flows and arise as limits of dilations of singularities of the geometric flows. Recently there have been a lot of studies of Ricci solitons and Yamabe solitons on compact or noncompact manifolds. These two solitons are special solutions of the Ricci flow and the Yamabe flow respectively and play important roles in the study of the singularities of these two flows. We can find results about the Ricci solitons and the Yamabe solitons in [2,3,7,8,11–13,17–19,22].

A τ-quasi Einstein metric is defined as

$$\mathrm{Ric}_{f,\tau}=\mathrm{Ric}+\nabla^2 f-\frac{\mathrm{d}f\otimes \mathrm{d}f}{\tau}=\lambda g,$$

where $\mathrm{Ric}_{f,\tau}$ is the τ-Bakry–Émery curvature, which always is used to replace the Ricci curvature when studying the weighted measure $\mathrm{d}\mu=e^{-f(x)}\,\mathrm{d}x$ [16], where $\mathrm{d}x$ is the Riemann–Lebesgue measure determined by the metric. Hence a quasi Einstein metric is the natural extension of a Ricci soliton. τ-quasi Einstein metrics are closely relative to the existence of warped product Einstein manifolds [1], which also have some different properties compared with the Ricci solitons. For example, it was proved in [14,20] that a τ-quasi Einstein metric with $\lambda>0$ is automatically compact when $\tau>0$ is finite. The complete results about τ-quasi Einstein metrics can be found in [17,18]. Similar to the τ-quasi Einstein metric, the τ-quasi Yamabe gradient soliton can be regarded as the natural extension of the Yamabe gradient soliton.

Let M be an n-dimensional connected Riemannian manifold. As defined in [10], for $\tau>0$, we call (M,g) a τ-quasi Yamabe gradient soliton if there exist a smooth potential function f on M and λ a constant such that

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$$\nabla^2 f - \frac{1}{\tau}\,\mathrm{d}f \otimes \mathrm{d}f = (R-\lambda)g, \tag{1.1}$$

here R denotes the scalar curvature of M. An ∞-quasi Yamabe gradient soliton implies a Yamabe gradient soliton. If $\lambda = 0$, $\lambda > 0$ or $\lambda < 0$, then the Yamabe gradient soliton is called Yamabe steady, Yamabe shrinking or Yamabe expanding, respectively.

It was proved in [8] that the scalar curvature of any compact Yamabe gradient soliton should be constant. Later the author of [9] gave a simple alternate proof for this result. By using the same method used in [9], the authors of [10] proved that the scalar curvature of any compact τ-quasi Yamabe gradient soliton should be constant. We will give an example to show that the scalar curvature of a noncompact quasi Yamabe gradient soliton is not necessary constant. But we can prove that the scalar curvature should be constant under an integral condition. We will do these in Section 2. Another result in Section 2 is the lower bound estimate for the scalar curvature.

There has been an active interest in the study of the classification for complete Yamabe gradient solitons. In [4], by using a result about complete conformal gradient soliton with nonnegative Ricci tensor, the authors proved that any complete noncompact Yamabe gradient soliton with positive Ricci tensor is rotationally symmetric, whenever the potential function is nonconstant. The authors of [12] proved that a Yamabe gradient soliton on a complete noncompact manifold has warped product structure in the region $\{|\nabla f| \neq 0\}$. They also got a non-local collapsing result for the Yamabe gradient soliton. In this paper, by almost the same method used in [12], we will show that a τ-quasi Yamabe gradient soliton also has warped product structure in the region $\{|\nabla f| \neq 0\}$. Gradient estimate is an important tool in geometric analysis [6,15,21]. Based on the gradient estimate for the potential function, we get a uniform lower bound for the injective radius, which implies a non-local collapsing result. We do these in Section 3.

In the last section, we first prove some estimates for the potential function when $\lambda \leqslant 0$, of independent interest. These estimates imply a finite topological type result for τ-quasi Yamabe gradient solitons with $\lambda < 0$. Moreover, we can get the decay estimate for $(R-\lambda)e^{-\frac{f}{\tau}}$ when $\lambda \leqslant 0$ provided with a Ricci pinching condition.

2. Properties of the scalar curvature

In [10], the authors proved that the scalar curvature of any compact τ-quasi Yamabe gradient soliton should be constant. In the following, we give a noncompact τ-quasi Yamabe gradient soliton with a nonconstant scalar curvature.

Example 2.1. We assume that $M = (0,+\infty) \times N^{n-1}$ is a warped product manifold with the product metric given by

$$\mathrm{d}s_M^2 = \mathrm{d}t^2 + \varphi^2(t)\,\mathrm{d}s_N^2,$$

where $\mathrm{d}s_N^2$ is a fixed metric on N and φ is a positive function on $(0,+\infty)$. Consider the orthonormal coframe $\{\theta_\alpha, 2 \leqslant \alpha \leqslant n\}$ on N^{n-1}, $\{\omega_1 = \mathrm{d}t,\ \omega_\alpha = \varphi(t)\theta_\alpha,\ 2 \leqslant \alpha \leqslant n\}$ is the orthonormal coframe on M^n. We use $R_{M,ijkl}$ and $R_{N,\alpha\beta\gamma\delta}$ to denote the Riemannian curvature tensors of M and N respectively. After the same calculation as in [18], we conclude that

$$R_{M,1\alpha ij} = \begin{cases} -(\log\varphi(t))'' - ((\log\varphi(t))')^2, & i = 1,\ j = \alpha, \\ (\log\varphi(t))'' + ((\log\varphi(t))')^2, & i = \alpha,\ j = 1, \\ 0, & \text{otherwise} \end{cases}$$

and

$$R_{M,\alpha\beta ij} = \begin{cases} \varphi^{-2}(t)R_{N,\alpha\beta\gamma\theta} + ((\log\varphi(t))')^2(\delta_{\alpha\theta}\delta_{\beta\gamma} - \delta_{\alpha\gamma}\delta_{\beta\theta}), & i = \gamma,\ j = \theta, \\ 0, & \text{otherwise.} \end{cases}$$

If we use $R_{N,\alpha\beta}$ to denote the Ricci curvature tensor on N, then the Ricci curvature tensor of M can be expressed as

$$R_{M,1i} = -(n-1)\big[\big(\log\varphi(t)\big)'' + \big(\big(\log\varphi(t)\big)'\big)^2\big]\delta_{1i}$$

and

$$R_{M,\alpha\beta} = \varphi^{-2}(t)R_{N,\alpha\beta} - \big[\big(\log\varphi(t)\big)'' + (n-1)\big(\big(\log\varphi(t)\big)'\big)^2\big]\delta_{\alpha\beta}.$$

For $\tau > 0$, we assume that

$$R_{N,\alpha\beta} = D_0\delta_{\alpha,\beta},$$

where $D_0 < n-2$ is a constant. Let $f(t,x) = f(t) = -2\tau\ln t$ with $\tau = \frac{(n-1)(n-2-D_0)}{2}$ and $\varphi(t) = t$. Note that $f_{11} = f''(t)$ and $f_{\alpha\alpha} = f'(t)(\ln\varphi)'(t)$. It is easy to verify that

$$R_M g = \mathrm{Hess}\, f - \frac{1}{\tau}\,\mathrm{d}f \otimes \mathrm{d}f$$