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Transplanting geometrical structures

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ABSTRACT

We say that a germ $\mathcal{G}$ of a geometric structure can be transplanted into a manifold M if there is a suitable geometric structure on M which agrees with $\mathcal{G}$ on a neighborhood of some point P of M. We show for a wide variety of geometric structures that this transplantation is always possible provided that M does in fact admit some such structure of this type. We use this result to show that a curvature identity which holds in the category of compact manifolds admitting such a structure holds for germs as well and we present examples illustrating this result. We also use this result to show geometrical realization problems which can be solved for germs of structures can in fact be solved in the compact setting as well.

1. Introduction

We shall consider the following geometric structures; precise definitions will be given in Section 2. We fix the dimension m of the underlying manifold and also the signature (p, q) where relevant.

Definition 1.1. Consider the possible structures:

(1) Affine structures.
(2) Pseudo-Riemannian structures of suitable signature.
(3) Almost (para)-complex structures in dimension $m = 2\bar{m}$.
(4) Almost (para)-Hermitian structures of suitable signature.
(5) (Para)-complex structures in dimension $m = 2\bar{m}$.
(6) (Para)-Hermitian structures of suitable signature.
(7) (Para)-Kähler structures of suitable signature.
(8) Weyl structures of suitable signature.
(9) (Para)-Kähler–Weyl structures of suitable signature.

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Let $\mathcal{S}$ be a structure from this list, let $\mathcal{G} \in \mathcal{S}$ be the germ of such a structure, and let M be a smooth manifold which admits a structure $S_M \in \mathcal{S}$. Fix a point $P \in M$. We say that $\mathcal{G}$ can be *transplanted* in M if there exists a structure $\tilde{S}_M \in \mathcal{S}$ such that $\tilde{S}_M$ near P is locally isomorphic to $\mathcal{G}$ and $\tilde{S}_M$ agrees with S_M away from P. We shall make this more precise in Section 2 – it is necessary to assume that M admits such a structure to avoid topological difficulties. For example, if $\mathcal{S}$ denotes the structure of Lorentzian metrics of signature $(1, m-1)$, then not every manifold will admit such a structure; similarly, if we are working with almost complex structures, not every manifold admits an almost complex structure.

Such problems arise in many contexts. One can often establish universal curvature identities for compact manifolds by considering the Euler–Lagrange equations of certain characteristic classes – see, for example, results in [10–13,17,26]. One then wants to show these identities hold more generally and this involves a transplanting problem. Similarly, when establishing geometric realization results, one often constructs examples which are only defined on a small neighborhood of the origin – the discussion in [4] provides a nice summary of these problems and we refer to other results in [5,6,15,16]. And one wants to then deduce these geometric realization results also hold in the compact setting.

1.1. Transplantation

The following is the first main result of this paper; a more precise statement for each of the structures in Definition 1.1 will be given subsequently in Section 2.

Theorem 1.1. *All the structures of Definition 1.1 can be transplanted.*

1.2. Curvature identities

Theorem 1.1 then yields the second main result of this paper which motivated our investigations in the first instance; we will present several applications in Section 3:

Theorem 1.2. *Let $\mathcal{S}$ be a structure of Definition 1.1. A curvature identity which holds for every $\xi \in \mathcal{S}(M)$ for M compact necessarily holds without the assumption of compactness.*

1.3. Geometric realizability

Theorem 1.1 and results described in [4] also yield the following result which formed part of the motivation of our paper; we will present several examples illustrating this result in Section 4:

Theorem 1.3. *Every algebraic model of the curvature tensor of a structure from Definition 1.1 is geometrically realizable by a compact manifold.*

2. The proof of Theorem 1.1

Notational conventions. We introduce some basic notational conventions that we shall employ throughout Section 2.

Definition 2.1. Let B_{3r} be the ball of radius $3r$ about the origin in $\mathbb{R}^m$ and let h be a smooth k-tensor defined on B_{3r}. Let $\|h\|_{3r}$ be the C^0 norm of h and let $\|h\|^1_{3r}$ be the C^1 norm of h on B_{3r}, i.e.

$$\|h\|_{3r} := \sup_{x\in B_{3r}, 1\leqslant i_1\leqslant m,\dots,1\leqslant i_k\leqslant m} \left|h_{i_1\dots i_k}(x)\right|,$$

$$\|h\|^1_{3r} := \|h\|_{3r} + \sup_{x\in B_{3r}, 1\leqslant j\leqslant m, 1\leqslant i_1\leqslant m,\dots,1\leqslant i_k\leqslant m} \left|\partial_{x_j} h_{i_1\dots i_k}(x)\right|.$$

We shall also denote these norms by $\|h\|$ and $\|h\|^1$ when no confusion is likely to result. We shall need $\|\cdot\|^1$ in Theorem 2.2 and Theorem 2.7 to be able to study geodesic completeness in Section 2.10; in the remaining results we will either study C^0 approximating transplants or simple transplants without estimates.

Definition 2.2. Let P_i be the base points of manifolds M_i. Let $\mathcal{S}$ be a structure from the list given in Definition 1.1. Let $S_1 \in \mathcal{S}(M_1)$ be the germ of a suitable structure on M_1 at P_1 and let $S_2 \in \mathcal{S}(M_2)$ be a structure which is defined on all of M_2. We will choose suitably normalized local coordinate systems $\vec{x} = (x^1, \dots, x^m)$ centered at P_1 which are defined on an open set $\mathcal{U}_1 \subset M_1$ and suitably normalized coordinates $\vec{y} = (y^1, \dots, y^m)$ centered at P_2 which are defined on an open set $\mathcal{U}_2 \subset M_2$. We use these coordinates to identify $\mathcal{U}_1$ and $\mathcal{U}_2$ with B_{3r} and $P_1 = P_2 = 0$ for some $r > 0$; we will often shrink r in the course of a particular discussion. We use the identification $\mathcal{U}_1 = B_{3r}$ to regard S_1 as defining a structure on B_{3r} and we use the identification $\mathcal{U}_2 = B_{3r}$ to regard S_2 as defining a structure on B_{3r} as well. Our task will be to find a structure $\tilde{S}_2 \in \mathcal{S}(B_{3r})$ so that $\tilde{S}_2 = S_1$ on B_r and so that $\tilde{S}_2 = S_2$ on B^c_{2r}; $\tilde{S}_2$ can then be extended to all of M_2 to agree with S_2 on $\mathcal{U}^c_2$. In this setting, we will say that $\tilde{S}_2$ *is isomorphic to* S_1 *near* P_2 *and agrees with* S_2 *away from* P_2.