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Sprays metrizable by Finsler functions of constant flag curvature

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ABSTRACT

In this paper we characterize sprays that are metrizable by Finsler functions of constant flag curvature. By solving a particular case of the Finsler metrizability problem, we provide the necessary and sufficient conditions that can be used to decide whether or not a given homogeneous system of second order ordinary differential equations represents the geodesic equations of a Finsler function of constant flag curvature. The conditions we provide are tensorial equations on the Jacobi endomorphism. We identify the class of homogeneous SODE where the Finsler metrizability is equivalent with the metrizability by a Finsler function of constant flag curvature.

1. Introduction

The inverse problem of Lagrangian mechanics can be formulated as follows: decide whether or not a given system of second order ordinary differential equations (SODE) coincides with the Euler–Lagrange equations of some Lagrangian, [2,6,11,15,17,19]. When the given system of SODE is homogeneous and the Lagrangian to search for is the square of a Finsler function, the problem is known as the Finsler metrizability problem, [8,14,18,23]. If the sought after Lagrangian is a Finsler function, the problem is known as the projective metrizability problem, or as the Finslerian version of Hilbert's fourth problem, [1,7–10,24].

In this paper we address the special case of the Finsler metrizability problem, where the Finsler function we seek for has constant curvature. When the spray has zero constant curvature, then there is no obstruction for the existence of a locally defined Finsler structure that metricizes the given spray, [7,9,18]. Therefore, in this work we will focus on the case when the curvature is non-zero.

In Theorem 4.1, we solve the above mentioned problem by providing a set of equations, which contains an algebraic equation A) and two tensorial differential equations D_1) and D_2) in (4.1), which have to be satisfied by the Jacobi endomorphism. One of these two tensorial equations restricts the class of homogeneous SODE (sprays), which we discuss, to the class of isotropic sprays. Therefore, we focus our attention on isotropic sprays and their relation with the Finsler

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metrizability problem. In Theorem 4.2, we characterize the class of isotropic sprays S for which the following conditions are equivalent:

- S is Finsler metrizable,
- S is metrizable by a Finsler metric of scalar flag curvature;
- S is Finsler metrizable by an Einstein metric;
- S is metrizable by a Finsler metric of constant flag curvature;
- S is Ricci constant.

If the Finsler function is reducible to a Riemannian metric, the equivalence of the above conditions, on manifolds of dimension grater than or equal to three, is always true for affine isotropic sprays, and it is not true in the general Finslerian context. Therefore, Theorem 4.2 identifies the class of sprays where this equivalence is still true. The last condition is a technical one, can be checked very easily, and it is common for both Theorems 4.1 and 4.2. Theorem 4.1 has the advantage of being true for spray spaces of dimension grater than or equal to two, and it has the disadvantage of disregarding Finsler metrizable sprays of non-constant flag curvature. The main advantage of Theorem 4.2 is that it treats sprays for which the Finsler metrizability is equivalent with the metrizability by a Finsler metric of constant non-zero flag curvature. The key ingredient in proving Theorem 4.2 is the Finslerian version of Schur's Lemma, [4, Lemma 3.10.2], which is true only for spray spaces of dimension grater than or equal to three.

Since any spray on a two-dimensional manifold is isotropic, one of the two equations (4.1) in Theorem 4.1 simplify. In Theorem 4.3 we provide necessary and sufficient conditions for the metrizability of a two-dimensional spray space by a Finsler function of constant (Gaussian) curvature.

To support our results, in the last section, we consider various examples of isotropic sprays that satisfy, or not, one or more of the necessary and sufficient conditions, which we provide, for Finsler metrizability.

2. Sprays and their geometric setting

The natural geometric framework for studying systems of second order ordinary differential equations is the tangent bundle of some configuration manifold.

In this work, M denotes a C^∞-smooth, real, and n-dimensional manifold. We will denote by TM its tangent bundle and by $T_0M = TM \setminus \{0\}$ the tangent bundle with the zero section removed. Local coordinate charts $(U, (x^i))$ on M induce local coordinate charts $(\pi^{-1}(U), (x^i, y^i))$ on TM, where $\pi : TM \to M$ is the canonical submersion. We will assume that M is a connected manifold of dimension $n \geqslant 2$. Therefore, TM and T_0M are $2n$-dimensional connected manifolds.

In this section we discuss the natural geometric setting determined by a spray S, which includes canonical nonlinear connection, dynamical covariant derivative and curvature tensors. This setting, as well as the proofs of our results in the next sections, are based on the Frölicher–Nijenhuis theory and the corresponding differential calculus that can be developed on TM, [11,13,22]. There are two canonical structures on TM, which we will use to develop our setting. One is the tangent structure, J, and the other one is the Liouville vector field, $\mathbb{C}$, locally given by

$$J = \frac{\partial}{\partial y^i} \otimes dx^i, \qquad \mathbb{C} = y^i \frac{\partial}{\partial y^i}.$$

A system of homogeneous second order ordinary differential equations on a manifold M, whose coefficients do not depend explicitly on time, can be identified with a special vector field on T_0M that is called a spray. A vector field $S \in \mathfrak{X}(T_0M)$ is called a *spray* if $JS = \mathbb{C}$ and $[\mathbb{C}, S] = S$. Locally, a spray S is given by

$$S = y^i \frac{\partial}{\partial x^i} - 2G^i(x, y) \frac{\partial}{\partial y^i}, \tag{2.1}$$

where functions $G^i(x, y)$ are smooth functions on domains of induced coordinates on T_0M and 2-homogeneous with respect to the y-variable.

It is well known that a spray induces a nonlinear connection, with the corresponding projectors h and v given by

$$h = \frac{1}{2}(\mathrm{Id} - \mathcal{L}_S J), \qquad v = \frac{1}{2}(\mathrm{Id} + \mathcal{L}_S J).$$

For a spray S, we consider the map $\nabla : \mathfrak{X}(T_0M) \to \mathfrak{X}(T_0M)$, given by [6]

$$\nabla = \mathcal{L}_S + h \circ \mathcal{L}_S h + v \circ \mathcal{L}_S v. \tag{2.2}$$

We require that the action of ∇ on scalar functions is given by $\nabla f = S(f)$, for $f \in C^\infty(T_0M)$. Further requirements that ∇ satisfies the Leibniz rule and commutes with contractions allow us to extend its action to arbitrary tensor fields on T_0M. We will refer to ∇ as to the *dynamical covariant derivative* induced by the spray S. Its action on semi-basic forms was called the *semi-basic derivation* and studied, in connection with the inverse problem of the calculus of variation, in [11].