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Approximation of rough functions

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Abstract

For given $p \in [1, \infty]$ and $g \in L^p(\mathbb{R})$, we establish the existence and uniqueness of solutions $f \in L^p(\mathbb{R})$, to the equation

$$f(x) - af(bx) = g(x),$$

where $a \in \mathbb{R}$, $b \in \mathbb{R} \setminus \{0\}$, and $|a| \neq |b|^{1/p}$. Solutions include well-known nowhere differentiable functions such as those of Bolzano, Weierstrass, Hardy, and many others. Connections and consequences in the theory of fractal interpolation, approximation theory, and Fourier analysis are established.

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1. Introduction

The subject of this paper, in broad terms, is fractal analysis. More specifically, it concerns a constellation of ideas centered around the single unifying functional equation (1). In practice,

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the given function $g(x)$ may be smooth and the solution $f(x)$ is often rough, possessing fractal features. Classical notions from interpolation and approximation theory are extrapolated, via this equation, to the fractal realm, the basic goal being the utilization of fractal functions to analyze real world rough data.

For given $p \in [1, \infty]$ and $g : \mathbb{R} \to \mathbb{R}$ with $g \in L^p(\mathbb{R})$, we establish the existence and uniqueness of solutions $f \in L^p(\mathbb{R})$, to the equation

$$f(x) - af(bx) = g(x), \tag{1}$$

where $a \in \mathbb{R}$, $b \in \mathbb{R} \setminus \{0\}$, and $|a| \neq |b|^{1/p}$. By uniqueness we mean that any solution is equal to f almost everywhere in $\mathbb{R}$. When a, b and g are chosen appropriately, solutions include the classical nowhere differentiable functions of Bolzano, Weierstrass, Hardy, Takagi, and others; see the reviews [2,13]. For example, the continuous, nowhere differentiable function presented by Weierstrass in 1872 to the Berlin Academy, defined by

$$f(x) = \sum_{k=0}^{\infty} a^k \cos(\pi b^k x), \tag{2}$$

where $0 < a < 1$, b is an integer, and $ab \geq 1 + \frac{3}{2}\pi$ (see [12]), is a solution to the functional equation (1) when $g(x) = \cos(\pi x)$. The graph of f was studied as a fractal curve in the plane by Besicovitch and Ursell [5]. An elementary and readable account of the history of nowhere differentiable functions is [23]; it includes the construction by Bolzano (1830) of one of the earliest examples of such a function. Analytic solutions to the functional equation (1) for various values of a and b, when g is analytic, have been studied by Fatou in connection with Julia sets [9,22]. If $g(x) = e^{\lambda x}$, then $f(x) = \sum_{k=0}^{\infty} a^k e^{b^k \lambda x}$ is a solution to Eq. (1) and is a special case of the Dirichlet series studied by Iserles and Wang [14] in the context of solutions to ordinary differential equations.

If $|ab| > 1$, $b > 1$ is an integer, and g has certain properties, see [2,13], then the graph of f, restricted to $[0, 1]$, has box-counting (Minkowski) dimension

$$D = 2 + \frac{\ln |a|}{\ln b}.$$

In particular, if $g(x) = \cos(\pi x)$, then by a recent result of Bárány, Romanowska, and Barański [1] the Hausdorff dimension of the graph of f is D, for a large set of values of $|a| < 1$.

Notation that is used in this paper is set in Section 2. In Section 3 we establish existence and uniqueness of solutions to Eq. (1) in various function spaces (see Theorem 1, Corollaries 1 and 4, Proposition 1). Although the emphasis has been on the pathology of the solution to the functional equation (1), it is shown that, if g is continuous, then the solution f is continuous (see Corollaries 2 and 3).

A widely used method for constructing fractal sets, in say $\mathbb{R}^2$, is as the attractor of an iterated function system (IFS). Indeed, starting in the mid 1980s, IFS fractal attractors A were systematically constructed so that A is the graph of a function $f : J \to \mathbb{R}$, where J is a closed bounded interval on the real line [3]. Moreover f can be made to interpolate the data $(x_0, y_0), (x_1, y_1), \ldots, (x_N, y_N)$, where $x_0 < x_1 < \cdots < x_N$ and $J = [x_0, x_N]$. The basic idea is to consider an IFS on $\mathbb{R}^2$ of the form $F = (\mathbb{R}^2; w_1, w_2, \ldots, w_N)$ where $w_n(x, y) = (L_n(x), F_n(x, y))$; L_n is a linear function that maps the interval J to the interval $[x_{n-1}, x_n]$; and w_n takes (x_0, y_0) to (x_{n-1}, y_{n-1}) and (x_N, y_N) to (x_n, y_n). Under appropriate conditions on the functions L_n and F_n (see Section 4 for details), there exists a unique closed