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Bernstein's Lethargy Theorem in Fréchet spaces

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Abstract

In this paper we consider Bernstein's Lethargy Theorem (BLT) in the context of Fréchet spaces. Let X be an infinite-dimensional Fréchet space and let $\mathcal{V} = \{V_n\}$ be a nested sequence of subspaces of X such that $\overline{V_n} \subseteq V_{n+1}$ for any $n \in \mathbb{N}$. Let e_n be a decreasing sequence of positive numbers tending to 0. Under one additional but necessary condition on $\sup\{\text{dist}(x, V_n)\}$, we prove that there exist $x \in X$ and $n_o \in \mathbb{N}$ such that

$$\frac{e_n}{3} \leq \text{dist}(x, V_n) \leq 3e_n$$

for any $n \geq n_o$. By using the above theorem, as a corollary we obtain classical Shapiro's (1964) and Tyuriemskih's (1967) theorems for Banach spaces. Also we prove versions of both Shapiro's (1964) and Tyuriemskih's (1967) theorems for Fréchet spaces. Considering rapidly decreasing sequences, other versions of the BLT theorem in Fréchet spaces will be discussed. We also give a theorem improving Konyagin's (2014) result for Banach spaces. Finally, we present some applications of the above mentioned result concerning particular classes of Fréchet spaces, such as Orlicz spaces generated by s-convex functions and locally bounded Fréchet spaces.

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1. Introduction

Let $C[0,1]$ denote the space of real valued continuous functions on $[0,1]$ with the supremum norm $\|.\|$. For $f \in C[0,1]$, the sequence of best approximations (or equivalently the distance from f to the linear subspace of polynomials P_n of degree $\leq n$) are defined as:

$$\operatorname{dist}(f, P_n) = \rho(f, P_n) = \rho_n(f) = \inf\{\|f - p\| : \ p \in P_n\}.$$

Clearly $\rho(f, P_1) \geq \rho(f, P_2) \geq \cdots$ and $\{\rho(f, P_n)\}$ form a decreasing sequence of best approximations. Furthermore we have,

(a) $\rho_n(\lambda f) = |\lambda| \rho_n(f)$
(b) $\rho_n(f + v) = \rho_n(f)$ for $f \in C[0,1]$ and $v \in P_n$
(c) $\rho_n(f_1 + f_2) \leq \rho_n(f_1) + \rho_n(f_2)$ and $|\rho_n(f_1) - \rho_n(f_2)| \leq \|f_1 - f_2\|$ for $f_1, f_2 \in C[0,1]$.

The last property implies the continuity of the mapping from $C[0,1]$ to $\mathbb{R}^+$ defined as $f \to \rho_n(f)$.

The density of polynomials in $C[0,1]$ implies that

$$\lim_{n\to\infty} \rho_n(f) = 0.$$

However, the Weierstrass approximation theorem gives no information about the speed of convergence for $\rho_n(f)$. Bernstein considered “the inverse problem of the theory of best approximation” and showed that for each decreasing, null sequence (d_n), there exists a function $f \in C[0,1]$ with $\rho_n(f) = d_n$, see [6]. This theorem is sometimes referred to as Bernstein's Lethargy Theorem (BLT) and it has been applied to the theory of quasianalytic functions in several complex variables [16] and used in the constructive theory of functions [22]. Following the proof of Bernstein, Timan [23] extended his result to an arbitrary system of strictly embedded *finite dimensional* subspaces Y_n. Later Shapiro, [21], replacing $C[0,1]$ with an arbitrary Banach space $(X, \|.\|)$ and the sequence of n-dimensional subspaces of polynomials of degree $\leq n$ by a sequence (Y_n) where $Y_1 \subset Y_2 \subset \cdots$ are strictly embedded *closed subspaces* of X, showed that in this setting, for each null sequence (d_n) of non-negative numbers, there is a vector $x \in X$ such that

$$\rho_n(x) = \inf\{\|x - u\| : \ u \in Y_n\} \neq O(d_n).$$

Thus, there is no $M > 0$ such that $\rho_n(x) \leq M d_n$ for all n. In other words, $\rho_n(x)$ can decay arbitrarily slowly. This result was strengthened by Tyuriemskih [25] who established that the sequence of best approximations may converge to zero at an arbitrary slow rate: for any expanding sequence $\{Y_n\}$ of subspaces and for any sequence $\{d_n\}$ of positive numbers converging to zero, he constructed an element $x \in X$ such that $\lim_{n\to\infty} \rho(x, Y_n) = 0$ and $\rho(x, Y_n) \geq d_n$ for all n. However, it is also possible that the sequence of best approximations may converge to zero arbitrarily fast. For example in [4], under same conditions imposed on $\{Y_n\}$ and $\{d_n\}$, for any sequence $\{c_n\}$ of positive numbers converging to zero, there exists an element $x \in X$ such that $\lim_{n\to\infty} \frac{\rho(x,Y_n)}{d_n} = 0$ but $\frac{\rho(x,Y_n)}{d_n} \neq O(c_n)$. For a generalization of Shapiro's theorem we refer the reader to [5]. For an application of Tyuriemskih's theorem to convergence of sequence of bounded linear operators consult [8]. For other versions of Bernstein's Lethargy Theorem see [1,2,4,12,18].

These ideas bring us to the following question: given a Banach space X, a strictly increasing sequence $\{Y_n\}$ of subspaces of X with a dense union, and a decreasing null sequence $\{d_n\} \subset [0,\infty)$, does there exists $x \in X$ such that $\operatorname{dist}(x, Y_n) = d_n$ for each n? The only known spaces X