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Approximation of additive random fields based on standard information: Average case and probabilistic settings

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ABSTRACT

We consider approximation problems for tensor product and additive random fields based on standard information in the average case setting. We also study the probabilistic setting of the mentioned problem for tensor products. The main question we are concerned with in this paper is "How much do we loose by considering standard information algorithms against those using general linear information?" For both types of the fields, the error of linear algorithms has been studied in great detail; however, the power of standard information was not addressed so far, which we do here. Our main result is that in most interesting cases there is no more than a logarithmic loss in approximation error when information is being restricted to the standard one. The results are obtained by randomization techniques.

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1. Introduction: general information against standard information

Let f be a function which we consider as an element of some Banach space $(B, \|\cdot\|)$ of functions. Assume that the whole function f is unknown but we are able to measure the values of some functionals $F_1(f), \ldots, F_n(f)$, such as values of f at certain points, its integrals, etc. Let $\Psi : \mathbb{R}^n \to B$ be some mapping. Then we may call

$$Af := \Psi(F_1(f), \ldots, F_n(f))$$

an *approximation algorithm* and define its error at f by $\|f - Af\|$. A typical problem setting assumes minimization of the approximation error for given n by optimizing the choice of the set $(F_j(\cdot))_{1\le j\le n}$ within available class of functionals, as well as the choice of the mapping Ψ.

As for the target function f, we essentially have two options for problem setting. Either we can let it vary over some set, typically some compact subset of B, for example, the unit ball of some other Banach space compactly embedded into B, by taking the *worst possible* approximation error for algorithm evaluation. Or we may consider f as a *random* function having certain distribution in B and using the expectation of the error for algorithm evaluation. The two possibilities are often referred to as the "worst case setting", resp. "average case setting". We will rather stick to the latter one and consider random functions (or *random fields*, if their arguments are multivariate) as the objects of approximation.

Let us now specify the problem furthermore. We will use a Hilbert norm for error evaluation, which essentially means $B = \mathbb{L}_2([0, 1]^d)$, where d is understood as appropriate parametric dimension of a random field we consider. It is well known that for Hilbert norms the minimal average error is attained on a *linear algorithm*, which means

$$Af = \sum_{j=1}^{n} F_j(f)\phi_j,$$

where every $F_j(\cdot)$ is a linear functional on B and $\phi_j \in B$ are specific fixed elements. It is often said that such algorithms are based on *linear information*.

Since this article deals only with Hilbert norms, we will only work with linear algorithms.

In many applications it is too costly to calculate values of arbitrary linear functionals needed for such approximation. It is therefore preferable to restrict the choice of $F_j(\cdot)$ by letting $F_j(f) := f(t_j)$, i.e. by taking values of f at some points. One says that such algorithms are based on *standard information*.

It is clear that algorithms based on linear information form a larger class than those based on standard information, thus, after optimization, the average error would be smaller in linear case. The main question we are concerned with in this paper is *"How much do we loose by considering standard information algorithms against those using general linear information?"*

This problem already received much interest, see [3,5,7,19] and especially the survey [14]. Most of the research was concentrated on the worst case setting. It was shown that in many cases the polynomial term of error decay (when considered as a function of the number n of functionals used) is the same for algorithms based on linear and standard information. However, there exist cases where the behavior of the error as a function of n for linear and standard information is drastically different. It means that the problem we consider is by far non-trivial.

In this paper we restrict investigation to one specific but very important class of averaging function distributions, or, equivalently, random fields under consideration. Namely we consider *tensor product random fields* and *additive random fields*. In short, tensor product random field on $[0, 1]^d$ is a zero mean random field with covariance function

$$K_d(s, t) := \prod_{l=1}^{d} K(s_l, t_l)$$

where $s = (s_1, \ldots, s_d), t = (t_1, \ldots, t_d) \in [0, 1]^d$, and $K(\cdot, \cdot)$ is a covariance function on $[0, 1]^2$. Brownian sheet, Brownian pillow, and other famous random fields belong to this class.