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Average sampling and space-frequency localized frames on bounded domains

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ABSTRACT

The primary goal of the paper is to develop average sampling on bounded domains in Euclidean spaces. As an application of this development we construct bandlimited and localized frames on domains and describe a scale of Besov spaces in terms of frame coefficients.

1. Introduction

Average sampling for bandlimited functions in $L_2(\mathbb{R})$ was initiated in [8] and then further developed in [2,1,3,19,20,25,26]. In the case of Riemannian manifolds without boundary the average sampling was introduced and analyzed in [21–24]. In this paper we consider average sampling for a particularly interesting class of manifolds with boundaries: bounded domains with smooth boundaries in Euclidean spaces. The corresponding spaces of "bandlimited" functions are defined as eigenspaces of an appropriate elliptic differential operator with Dirichlet boundary conditions. We use our development to construct bandlimited and localized frames on bounded domains.

Let $\Omega \subset \mathbb{R}^d$ be a domain with a smooth boundary Γ. In the space $L_2(\Omega)$ we consider a strictly elliptic self-adjoint positive definite operator L generated by an expression

$$Lf = -\sum_{k,i=1}^{d} \partial_{x_k}(a_{k,i}(x)\partial_{x_i} f), \tag{1.1}$$

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with zero boundary condition. This way we obtain a self-adjoint positive definite operator in the Hilbert space $L_2(\Omega)$ with a discrete spectrum $0 < \lambda_1 \leq \lambda_2 \leq \cdots$, which goes to infinity. Let $\{u_j\}$ be a corresponding set of eigenfunctions which forms an orthonormal basis of $L_2(\Omega)$.

Definition 1. The notation $E_{[\sigma,\omega]}(L)$, $0 \leq \sigma < \omega$, will be used for the span of all eigenfunctions u_j whose corresponding eigenvalues λ_j belong to $[\sigma, \omega]$. It will be called the space of functions bandlimited to $[\sigma, \omega]$. In the case $\sigma = 0$ the notation $E_\omega(L)$ will be used.

In Section 3 we develop average sampling in spaces of bandlimited functions $E_\omega(L)$, $\omega > 0$. As a first step we prove in Lemma 3.1 a generalized version of the Poincaré inequality. Namely, it is shown that if $U \subset Q(\rho)$ where $Q(\rho)$ is a standard cube of diameter ρ and $d\mu$ is a positive measure on the set U then for any f in the Sobolev space $H^m(Q(2\rho))$, $m > d/2$, one has

$$\left\| f - \frac{1}{|U|} \int_U f d\mu \right\|^2_{L_2(U)} \leq C \sum_{1 \leq |\alpha| \leq m} \rho^{2\alpha} \|\partial^\alpha f\|^2_{L_2(Q(2\rho))}, \qquad |U| = \int_U d\mu, \tag{1.2}$$

where C is independent on f; $\alpha = (\alpha_1, \ldots, \alpha_d)$, $\partial^\alpha f = \partial^{\alpha_1}_{x_1} \cdots \partial^{\alpha_d}_{x_d} f$ is a partial derivative of order $|\alpha| = \alpha_1 + \cdots + \alpha_d$, and $L_2(U) = L_2(U, dx)$ where dx is the regular Lebesgue measure.

Remark 1.1. Note that in the case of the regular Poincaré inequality one has

$$\left\| f - \frac{1}{|U|} \int_U f dx \right\|^2_{L_2(U)} \leq C \|\nabla f\|^2_{L_2(U)}, \quad f \in H^1(U),$$

i.e. there are just first derivatives on the right side of the corresponding inequality. However, it is impossible to reduce the order of the derivatives in our case since our inequality includes, for example, the Dirac measure $d\mu = \delta_x$, $x \in U$, which requires at least continuity of the function f and it is guaranteed by the condition $m > d/2$.

Next, we prove in Theorem 3.4 that every function in a subspace $E_\omega(L)$, $\omega > 0$ is uniquely determined by its average values over "small" subsets "uniformly" distributed over the domain Ω. This fact leads to a construction of an "almost tight" Hilbert frame in each subspace $E_\omega(L)$, $\omega > 0$.

It is important to note that in Lemma 3.3 which provides discretization of the norm in a space $E_\omega(L)$ the number of "samples" $\Phi_i(f)$ is approximately $Vol(\Omega)\omega^{d/2}$, which according to Weyl's asymptotic formula is essentially the dimension of the space $E_\omega(L)$ because (see [15])

$$dim\, E_\omega(L) \sim Vol(\Omega)\omega^{d/2}.$$

In this sense Lemma 3.3 is optimal.

In Section 4 we construct a set of projectors (not orthogonal)

$$F_j : L_2(\Omega) \to E_{[2^{2j-2}, 2^{2j+2}]}(L)$$

such that for every $f \in L_2(L)$

$$\|f\|^2 = \sum_{j \in \mathbb{N}} \|F_j f\|^2.$$

By applying Theorem 3.4 we obtain existence of an "almost tight" bandlimited frame in $L_2(\Omega)$ (Theorem 4.1).

Moreover, in Section 5 we prove that every function in our frame has very strong localization on the manifold (Theorem 5.2).

Thus we construct a frame $\{\varphi_{j,i}\}$, $j = 0, 1, 2, \ldots$, $i = 1, 2, \ldots, I_j$, in the space $L_2(\Omega)$ which has the following distinguished properties:

1. frame constants are close to one;
2. every frame member $\varphi_{j,i}$ is bandlimited, i.e. $\varphi_{j,i} \in E_{[2^{2j-2}, 2^{2j+2}]}(L)$;
3. if $j \in \mathbb{N}$ is large enough then frame function $\varphi_{j,i}$ looks "almost" like a delta function.