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Fast orthogonal transforms and generation of Brownian paths

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ABSTRACT

We present a number of fast constructions of discrete Brownian paths that can be used as alternatives to principal component analysis and Brownian bridge for stratified Monte Carlo and quasi-Monte Carlo. By fast we mean that a path of length n can be generated in $O(n\log(n))$ floating point operations. We highlight some of the connections between the different constructions and we provide some numerical examples.

1. Orthogonal transforms and Brownian paths

There are several constructions that are frequently used to construct *discrete Brownian paths*, by which we mean a random function B on a given set $\{t_1, \ldots, t_n\} \subseteq \mathbb{R}$, $0 < t_1 < \cdots < t_n \leq 1$, so that $B = (B_{t_1}, \ldots, B_{t_n})$ is a Gaussian vector with mean zero and covariance matrix

$$\left(\min(t_j, t_k)\right)_{j,k=1}^{n} = \begin{pmatrix} t_1 & t_1 & t_1 & \ldots & t_1 \\ t_1 & t_2 & t_2 & \ldots & t_2 \\ t_1 & t_2 & t_3 & \ldots & t_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ t_1 & t_2 & t_3 & \ldots & t_n \end{pmatrix}.$$

The case where the t_j are evenly spaced is the most important one from the practical point of view. In that case the covariance matrix equals

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$$\left(\frac{1}{n}\min(j,k)\right)_{j,k=1}^{n} = \frac{1}{n}\begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & 2 & 2 & \dots & 2 \\ 1 & 2 & 3 & \dots & 3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 2 & 3 & \dots & n \end{pmatrix}.$$

Throughout the paper this matrix will be denoted by $\Sigma^{(n)}$ or, if there is no danger of confusion, simply by Σ.

The arguably most straightforward method (a.k.a. *forward method* or *step-by-step method*) is to compute the cumulative sum of n independent normal variables of mean zero and variance $\frac{1}{n}$. All constructions we present in this article are equivalent to this simple method from the probabilistic point of view.

However, there are refined simulation methods, for example stratified sampling (cf. [10]) and quasi-Monte Carlo methods (see [19]), which achieve higher convergence rates for some problems. Those techniques have in common that they require the identification of more important and less important input variables. For many problems the straightforward method does not provide this.

For this reason alternatives to the forward construction are frequently used, the Brownian bridge (BB) construction (a.k.a. Lévy–Ciesielski construction or midpoint displacement) and the principal component analysis (PCA) construction (a.k.a. singular value construction). The first use of the BB construction in finance is due to [18], the first use of PCA for financial applications was presented in [2], both with dramatic improvement of convergence rates.

It has been mentioned by Papageorgiou [20] that in fact any decomposition $AA^T = \Sigma$ provides a construction for a discrete approximation of a Brownian path via $Y = AX$, where X is a standard normal vector. In that context, the forward construction corresponds to the Cholesky decomposition of Σ, $\Sigma = SS^\top$, where S is the summation operator

$$S = \frac{1}{\sqrt{n}}\begin{pmatrix} 1 & 0 & \dots & 0 \\ 1 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 1 \end{pmatrix}. \tag{1}$$

PCA corresponds to $A = VD$, where $\Sigma = VD^2V^\top$ is the singular value decomposition of Σ. A corresponding decomposition for the BB algorithm is given, for example, in [15].

However, Papageorgiou [20] notes that there are examples where BB and PCA are not giving better results than the forward method in connection with quasi-Monte Carlo. He further shows that the worst case error of integration for a certain class of payoff functions is independent of the path construction. Thus in the sense of worst case error all decompositions are equivalent.

This has been investigated further by Sloan and Wang [24]. The authors show another equivalence principle which roughly states that every decomposition is equally bad and good for QMC, depending on the function that one wants to integrate. For every decomposition A that is good for one payoff function f, and every decomposition $\tilde{A}$ there is another payoff function $\tilde{f}$ for which $\tilde{A}$ is equally good.

It is therefore prudent to tailor the decomposition to the problem at hand. This is done, for example, by Imai and Tan [12].

While the possible decompositions of Σ provide a clean framework for the study of algorithms for generation of Brownian paths, they are of limited practical value because the matrix-vector multiplication is comparatively slow for all but very small values of n. This is the case since general matrix-vector multiplication uses $O(n^2)$ floating point operation (*flops*), while the forward method and the Brownian bridge use only $O(n)$ flops.

Until recently this has been considered a serious disadvantage of PCA as well, cf. [10]. Yet it has been shown by Scheicher [23], using results from Åkesson and Lehoczky [1], that PCA can be computed using the fast sine transform, thereby using $O(n\log(n))$ flops.

While the importance of the proper choice of the decomposition $AA^\top = \Sigma$ for problems arising from quasi-Monte Carlo pricing of financial derivatives is stressed by a number of authors, see e.g. [24,12], there has been a lack of alternatives to the three aforementioned constructions that