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Classification of global phase portraits and bifurcation diagrams of Hamiltonian systems with rational potential

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Abstract

In this paper we study the global dynamics of the Hamiltonian systems $\dot{x} = H_y(x, y)$, $\dot{y} = -H_x(x, y)$, where the Hamiltonian function H has the particular form $H(x, y) = y^2/2 + P(x)/Q(x)$, $P(x), Q(x) \in \mathbb{R}[x]$ are polynomials, in particular H is the sum of the kinetic and a rational potential energies. Firstly, we provide the normal forms by a suitable μ-symplectic change of variables. Then, the global topological classification of the phase portraits of these systems having canonical forms in the Poincaré disk in the cases where degree$(P) = 0, 1, 2$ and degree$(Q) = 0, 1, 2$ are studied as a function of the parameters that define each polynomial. We use a blow-up technique for finite equilibrium points and the Poincaré compactification for the infinite equilibrium points. Finally, we show some applications.

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1. Introduction and statement of the main results

We consider a rational Hamiltonian vector field on $\mathbb{R}^2$, that is, an autonomous Hamiltonian function with 1-degree of freedom of the form

$$\dot{x} = H_y(x, y), \quad \dot{y} = -H_x(x, y), \tag{1}$$

with Hamiltonian function

$$H = H(x, y) = \frac{y^2}{2} + V(x), \tag{2}$$

where the potential

$$V(x) = \frac{P(x)}{Q(x)}, \tag{3}$$

is such that $P(x), Q(x) \in \mathbb{R}[x]$ are polynomials with real coefficients of degrees n and m, respectively, which do not have a common factor, i.e., they are coprime. The set $\mathcal{S} = \{(x, y) : Q(x) = 0\}$ represents the singularities of the vector field (1) (i.e., the points where the vector field (1) is not defined). Considering $P = \sum_{i=0}^{n} a_i x^i$, $Q = \sum_{i=0}^{m} b_i x^i$, $a_i, b_i \in \mathbb{R}$, the Hamiltonian function $H(x, y)$ depends on $n + m + 2$ parameters and it is analytic on $\Omega = \mathbb{R}^2 \setminus \mathcal{S}$.

The Hamiltonian system associated to (2) has the form

$$\dot{x} = y, \quad \dot{y} = -\frac{[P'(x)\, Q(x) - P(x)\, Q'(x)]}{Q^2(x)}, \tag{4}$$

where $P'(x)$, $Q'(x)$ indicates the derivatives of P and Q with respect to x. We will consider the cases where the polynomials $P(x)$ and $Q(x)$ have degree 0, 1 or 2, and in this work we will classify the global phase portraits of the mechanical vector fields defined in (4) as function of the parameters associated to the polynomials P and Q on the Poincaré disk.

The dynamical behavior of the polynomial Hamiltonian planar vector fields is being studied currently, see for example [1,8,9,2–4]. In these last four works the global dynamics are characterized in global phase portraits, assuming that the linear part has a center. The perturbation of Hamiltonian systems with a center is also a topic of study, see [13] and references therein. In [11] it was considered a separable Hamiltonian $H(x, y) = F(x) + G(y)$ and an algorithm to obtain the phase portraits.

In our approach, the first step is to transform the rational vector field into (4) in a polynomial vector field. In fact, by considering the re-scaling in the time, by means

$$\frac{dt}{d\tau} = Q^2(x) > 0, \tag{5}$$

the rational Hamiltonian vector field (1) becomes the special polynomial vector field

$$x' = yQ^2(x) \equiv P_1(x, y), \quad y' = -[P'(x)\, Q(x) - P(x)\, Q'(x)] \equiv P_2(x), \tag{6}$$

where $()'$ denotes derivative with respect to τ. We emphasize that these systems have a first integral of motion, given by the function H defined in (2), and in general, the new system