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An inverse blow-up problem

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Abstract

This paper studies an inverse problem to determine a nonlinearity of an autonomous equation from blow-up time of solutions to the equation. Firstly we prove a global continuation result showing that a nonlinearity realizing blow-up time for large initial data can be continued in the direction of smaller data as long as the blow-up time is Lipschitz continuous. Secondly we develop a method based upon a Wiener–Hopf structure by which the existence and uniqueness of a nonlinearity realizing blow-up time for large initial data is shown. These enable us to establish a global existence and uniqueness result for the inverse problem.

MSC: 34A55; 45G05

Keywords: Inverse problem; Blow-up time; Nonlinear integral equation

1. Introduction and main results

Does a data-set of blow-up time of solutions to a super-linear differential equation determine the nonlinearity of the equation? This is the subject of this paper. Though there might be much possibility of choices of differential equations as well as initial conditions that we may treat on this subject, we confine ourselves, in this paper, to the following simple system:

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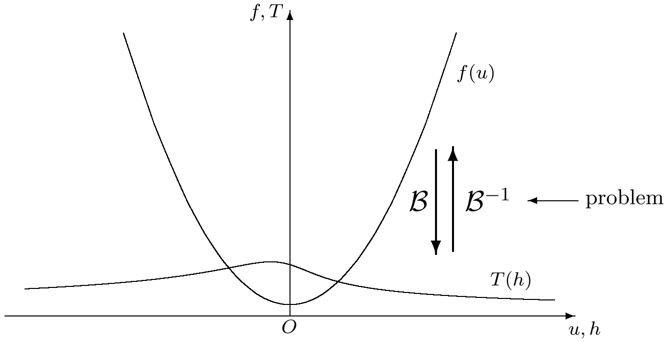

Fig. 1. Inverse blow-up problem.

$$\begin{cases} \dfrac{d^2u}{dt^2} = f(u), & 0 < t < \infty; \\ u(0) = h, & a_0 < h < \infty; \\ \dfrac{du}{dt}(0) = 0. \end{cases} \tag{1.1}$$

Here $-\infty \le a_0 < \infty$ and f is a positive, continuous function on the interval (a_0, ∞). Suppose that f is super-linear in the sense that f satisfies the condition

$$\int_a^\infty \frac{du}{\sqrt{\int_a^u f(\xi)d\xi}} < \infty, \tag{1.2}$$

for each $a > a_0$. Then the solution $u = u(t, h)$ of (1.1) for each $h \in (a_0, \infty)$ blows up at the time

$$T_f(h) := \frac{1}{\sqrt{2}} \int_h^\infty \frac{du}{\sqrt{\int_h^u f(\xi)d\xi}} \tag{1.3}$$

for each $h \in (a_0, \infty)$ (see [5]). We hereafter call $T = T_f(h)$ the blow-up time function for f and let $\mathcal{B}$ denote a map assigning T_f to f, that is $\mathcal{B} : f \to T_f$. Then the problem we discuss in this paper can be stated as

Problem 1.1 *(Inverse blow-up problem).* Given a positive function $T = T(h)$ on (a_0, ∞), determine a positive nonlinearity f in (1.1) so that $\mathcal{B}f = T$.

This inverse problem is to consider the inverse $\mathcal{B}^{-1}$ of the blow-up time map $\mathcal{B}$ (see Fig. 1), which contains the following fundamental issues. (1) Given $T = T(h)$ on (a_0, ∞), does there exist a positive solution f on (a_0, ∞) of $\mathcal{B}f = T$? (2) Is f unique for each T? Actually, in Fig. 1, we give a portrait of the blow-up time function for $f(u) = u^2 + 1$ and the blow-up time function $T(h)$ corresponding to it. Even in this simple example, issue (2), namely, question whether this $T(h)$ is realized only by $f(u) = u^2 + 1$ is not trivial at all. Problem 1.1 involves such a basic question.