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Global well-posedness of the 2D nonhomogeneous incompressible nematic liquid crystal flows ☆

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Abstract

This paper concerns the Cauchy problem of the two-dimensional (2D) nonhomogeneous incompressible nematic liquid crystal flows on the whole space $\mathbb{R}^2$ with vacuum as far field density. It is proved that the 2D nonhomogeneous incompressible nematic liquid crystal flows admit a unique global strong solution provided that the initial data density and the gradient of orientation decay not too slow at infinity, and the initial orientation satisfies a geometric condition (see (1.3)). In particular, the initial data can be arbitrarily large and the initial density may contain vacuum states and even have compact support. Furthermore, the large time behavior of the solution is also obtained.

MSC: 76A15; 35B65; 35Q35

Keywords: Nonhomogeneous incompressible nematic liquid crystal flow; Strong solutions; Vacuum; Temporal decay

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1. Introduction

Liquid crystals can form and remain in an intermediate phase of matter between liquids and solids. When a solid melts congruently, if the energy gain is enough to overcome the positional order but the shape of the molecules prevents the immediate collapse of orientational order, liquid crystals are formed. The lack of positional order is a shared property of liquid crystals and liquids; on the other hand, liquid crystals are anisotropic (like solids). The nematic liquid crystals exhibit long-range ordering in the sense that their rigid rod-like molecules arrange themselves with their long axes parallel to each other. Their molecules float around as in a liquid, but have the tendency to align along a preferred direction due to their orientation. The hydrodynamic theory of the nematic liquid crystals was first derived by Ericksen and Leslie during the period of 1958 through 1968 (see [8,20]). A brief account of the Ericksen–Leslie theory on nematic liquid crystal flows and the derivations of several approximate systems can be found in the appendix of [29]. For more details on the hydrodynamic continuum theory of liquid crystals, we refer the readers to the book of Stewart [41].

In this paper, we investigate the global existence of solutions of the following two dimensional (2D) simplified version of nematic liquid crystal flow in the whole space $\mathbb{R}^2$, which describes the motion of a nonhomogeneous incompressible flow of nematic liquid crystals (see, e.g., [27,28]):

$$\begin{cases} \varrho_t + \operatorname{div}(\varrho u) = 0, \\ \varrho u_t + \varrho u \cdot \nabla u - \nu \Delta u + \nabla P = -\lambda \operatorname{div}(\nabla d \odot \nabla d), \\ d_t + (u \cdot \nabla) d = \gamma(\Delta d + |\nabla d|^2 d), \\ \quad \operatorname{div} u = 0, \qquad |d| = 1, \end{cases} \tag{1.1}$$

where $\varrho(x,t) : \mathbb{R}^2 \times (0,+\infty) \to \mathbb{R}$ is the density, $u(x,t) : \mathbb{R}^2 \times (0,+\infty) \to \mathbb{R}^2$ is the unknown velocity field of the flow, $P(x,t) : \mathbb{R}^2 \times (0,+\infty) \to \mathbb{R}$ is the scalar pressure and $d : \mathbb{R}^2 \times (0,+\infty) \to \mathbb{S}^2$, the unit sphere in $\mathbb{R}^3$, is the unknown (averaged) macroscopic/continuum molecule orientation of the nematic liquid crystal flow, $\operatorname{div} u = 0$ represents the incompressible condition, and ν, λ and γ are positive numbers associated with the properties of the material: ν is the kinematic viscosity, λ is the competition between kinetic energy and potential energy, and γ is the microscopic elastic relaxation time for the molecular orientation field. The notation $\nabla d \odot \nabla d$ denotes the 2×2 matrix whose (i,j)-th entry is given by $\partial_i d \cdot \partial_j d$ $(1 \le i, j \le 2)$.

We consider the Cauchy problem for (1.1) with initial conditions

$$\varrho(x,0) = \varrho_0(x), \quad \varrho u(x,0) = \varrho_0 u_0(x), \quad d(x,0) = d_0(x), \quad |d_0(x)| = 1, \quad \text{in } \mathbb{R}^2 \tag{1.2}$$

for given initial data ϱ_0, u_0, and d_0. Since the concrete values of ν, λ and γ do not play a special role in our discussion, for simplicity, we assume that they are all equal to one throughout this paper.

The above system (1.1)–(1.2) is a simplified version of the Ericksen–Leslie model [8,20], but it still retains most important mathematical structures as well as most of the essential difficulties of the original Ericksen–Leslie model. Mathematically, system (1.1)–(1.2) is a strongly coupled system between the nonhomogeneous incompressible Navier–Stokes equations (see, e.g., [26, 34,38,43]) and the transported heat flows of harmonic map (see, e.g., [2,18,44]), and thus, its mathematical analysis is full of challenges.