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Entire solutions of the degenerate Monge–Ampère equation with a finite number of singularities ☆

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Abstract

We determine the global behavior of every $\mathcal{C}^2$-solution to the two-dimensional degenerate Monge–Ampère equation, $u_{xx}u_{yy} - u_{xy}^2 = 0$, over the finitely punctured plane. With this, we classify every solution in the once or twice punctured plane. Moreover, when we have more than two singularities, if the solution u is not linear in a half-strip, we obtain that the singularities are placed at the vertices of a convex polyhedron P and the graph of u is made by pieces of cones outside of P which are suitably glued along the sides of the polyhedron. Finally, if we look for analytic solutions, then there is at most one singularity and the graph of u is either a cylinder (no singularity) or a cone (one singularity).

MSC: 35K10; 53C21; 53A05

1. Introduction

A celebrated result proved by A.V. Pogorelov [16], and independently by P. Hartman and L. Nirenberg [8], states that all the global solutions to the degenerate Monge–Ampère equation

$$u_{xx}u_{yy} - u_{xy}^2 = 0, \tag{1}$$

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where $u : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is a function of class $\mathcal{C}^2$, are given by

$$u(x, y) = \alpha(x) + c_0\, y,$$

up to a rotation in the (x, y)-plane, that is, the graph of u is a cylinder (see also [18,19]).

This degenerate Monge–Ampère equation has been extensively studied from an analytic point of view and also from a geometric point of view since the graph of every solution determines a flat surface in the Euclidean 3-space.

Local properties of the solutions of (1) have been analyzed in many papers (see, for instance, [20,21] and references therein). Our objective is to study the global behavior of the solutions of (1) for the largest non-simply-connected domains, that is, in the finitely punctured plane.

Observe that, when the Monge–Ampère equation is elliptic or hyperbolic, a large amount of work in the understanding of these solutions in the punctured plane has been achieved from a local and global point of view (see, among others, [1–7,9–11,13,14,17]).

The paper is organized as follows. After a first section of preliminaries, in Section 3 we establish the global behavior of every $\mathcal{C}^2$ solution $u : \mathbb{R}^2\backslash\{p_1, \dots, p_n\} \longrightarrow \mathbb{R}$ to the Monge–Ampère equation (1) with isolated singularities at the points p_i. We show that for every singular point p_i there exists at least a sector $S_i \subseteq \mathbb{R}^2\backslash\{p_1, \dots, p_n\}$ such that the graph of u over S_i is a piece of a cone. Moreover, there exists at most a maximal strip S such that the graph of u over S is a cylinder. As a consequence of these results, we obtain that if u is an analytic solution then there is at most one singularity, and the graph of u is either a cylinder if there is no singularity or a cone if there is one singularity.

In Section 4 we classify all solutions to (1) with one or two singularities. In particular, if u is a solution with one singularity at p_0 then, either the graph of u is a cone or there exists a line r containing to p_0 such that the graph of u is a cylinder over one half-plane determined by r and a cone over the other half-plane. We also describe every solution in the twice punctured plane.

Although the behavior of a solution u to (1) with more than two singularities can be complicated, due to the existence of some regions in the (x, y)-plane where u is a linear function, we classify in Section 5 every solution which does not admit a large region where u is linear, that is, every solution such that the domain where u is linear does not contain a half-strip of $\mathbb{R}^2$. In such case, we show that the singular points are the vertices of a convex compact polyhedron $C \subseteq \mathbb{R}^2$ with non-empty interior, the solution u must be linear over C and the graph of u is made by some cones over $\mathbb{R}^2\backslash C$ which are suitably glued along the segments of the boundary of the planar polyhedron $u(C)$.

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2. Preliminaries

Let $\Omega \subseteq \mathbb{R}^2$ be a domain, and $u : \Omega \longrightarrow \mathbb{R}$ be a function of class $\mathcal{C}^2$ satisfying the degenerate Monge–Ampère equation $u_{xx}u_{yy} - u_{xy}^2 = 0$, in Ω. As we mentioned in the introduction, it is well known that every solution $u(x, y)$ to the degenerate Monge–Ampère equation in the whole plane $\mathbb{R}^2$ is given by $u(x, y) = \alpha(x) + c_0\, y$, up to a rotation in the (x, y)-plane (see [8,12,16]), that is, its graph is a cylinder.

Thus, it is natural to study the solutions to the degenerate Monge–Ampère equation in the possible largest domains. In other words, we consider solutions to (1) in the non-simply-connected