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Navier–Stokes flow in the weighted Hardy space with applications to time decay problem

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Abstract

The asymptotic expansions of the Navier–Stokes flow in $\mathbb{R}^n$ and the rates of decay are studied with aid of weighted Hardy spaces. Fujigaki and Miyakawa [12], Miyakawa [28] proved the nth order asymptotic expansion of the Navier–Stokes flow if initial data decays like $(1 + |x|)^{-n-1}$ and if nth moment of initial data is finite. In the present paper, it is clarified that the moment condition for initial data is essential in order to obtain higher order asymptotic expansion of the flow and to consider the rapid time decay problem. The second author [39] established the weighted estimates of the strong solutions in the weighted Hardy spaces with small initial data which belongs to L^n and a weighed Hardy space. Firstly, the refinement of the previous work [39] is achieved with alternative proof. Then the existence time of the solution in the weighted Hardy spaces is characterized without any Hardy norm. As a result, in two dimensional case the smallness condition on initial data is completely removed. As an application, the rapid time decay of the flow is investigated with aid of asymptotic expansions and of the symmetry conditions introduced by Brandolese [3].

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Keywords: Navier–Stokes equations; Weighted Hardy spaces; Asymptotic expansion; Time decay

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1. Introduction

Let $n \geq 2$. We consider the Navier–Stokes equations in $\mathbb{R}^n$ which describe the motion of the incompressive viscous fluid:

$$\begin{cases} \partial_t u - \Delta u + u \cdot \nabla u + \nabla \Pi = 0 \quad \text{in } \mathbb{R}^n \times (0, \infty), \\ \operatorname{div} u = 0 \quad \text{in } \mathbb{R}^n \times (0, \infty), \\ u(x, 0) = a(x) \quad \text{in } \mathbb{R}^n, \end{cases} \tag{N–S}$$

where $u(x,t) = \big(u_1(x,t), \dots, u_n(x,t)\big)$ and $\Pi(x,t)$ denote the unknown velocity and the pressure of the fluid at $(x,t) \in \mathbb{R}^n \times (0, \infty)$, respectively, while $a(x) = \big(a_1(x), \dots, a_n(x)\big)$ denotes the given initial data.

Since the celebrated paper of Leray's [21], the time decay problem is one of main interests in the mathematical fluid mechanics. Firstly, Masuda [23] gave a partial answer to the energy decay for weak solutions. M. Schonbek [32,33] derived the explicit rate of the algebraic time decay, introducing, so-called, the Fourier splitting method. Kajikiya and Miyakawa [15], Wiegner [40, 41] obtained the upper bound

$$\|u(t)\|_2 \leq C(1+t)^{-\frac{n+2}{4}}, \tag{1.1}$$

if a satisfies, for instance,

$$\int_{\mathbb{R}^n} (1+|x|)|a(x)|\,dx < \infty. \tag{1.2}$$

See also, [34,35]. It is clarified that (1.1) is optimal decay for the general Navier–Stokes flow whose initial data satisfies (1.2). Indeed, Carpio [8], Fujigaki and Miyakawa [12] derived the asymptotic expansion of the Navier–Stokes flow under (1.2). Furthermore, in terms of the coefficients of the first order expansion of the Navier–Stokes flow, Miyakawa and Schonbek [31] formulated necessary and sufficient condition for $\|u(t)\|_2 = o(t^{-\frac{n+2}{4}})$ as $t \to \infty$. Although Fujigaki and Miyakawa [12] showed higher order expansions of the Navier–Stokes flow, they need more stringent condition, i.e., pointwise decay on initial data such as, for instance, $|a(x)| \leq C(1+|x|)^{-n-1}$ besides the moment condition. See also [28,30]. Later, Kukavica and Reis [19] derived the general order expansions for a in the Schwartz class. However, it seems that there are few results which deal with unbounded initial data for higher order asymptotic expansions and rapid time decay compared with (1.1).

The aim of this paper is to establish weighted estimates and mth order asymptotic expansions of the Navier–Stokes flow for $m = 1, \dots, n$ with only the moment condition on initial data, which allowed us to deal with unbounded initial data. Furthermore, we derive the rapid time decay assuming the symmetry of the flow, which is introduced by Brandolese [3]. In other words, the moment condition of initial data is enough to obtain higher order expansion and also rapid time decay. For the weighted estimates and the rapid time decay, see [14,38,6,1,36,2,5,20,9,18,17].

For this purpose, we firstly introduce the weighted Hardy space. The second author [39] proved the global well-posedness in weighted Hardy spaces for small initial data. Theory of real