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Strongly coupled elliptic equations related to mean-field games systems

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Abstract

In this paper, we study existence of solutions for the following elliptic problem, related to mean-field games systems:

$$\begin{cases} -\mathrm{div}(M(x)\nabla\zeta) + \zeta - \mathrm{div}(\zeta\, A(x)\nabla u) = f & \text{in } \Omega, \\ -\mathrm{div}(M(x)\nabla u) + u + \theta\, A(x)\nabla u \cdot \nabla u = \zeta^p & \text{in } \Omega, \\ \zeta = 0 = u & \text{on } \partial\Omega, \end{cases}$$

where $p > 0$, $0 < \theta < 1$, and $f \geq 0$ is a function in some Lebesgue space.

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1. Introduction

Let Ω be a bounded, open subset of $\mathbb{R}^N$, $N \geq 2$, and let $M : \Omega \to \mathbb{R}^{N^2}$ and $A : \Omega \to \mathbb{R}^{N^2}$ be matrices such that

$$M(x)\xi \cdot \xi \geq \alpha|\xi|^2, \quad |M(x)| \leq \beta, \tag{1.1}$$

and

$$A(x)\xi \cdot \xi \geq \alpha|\xi|^2, \quad |A(x)| \leq \beta, \tag{1.2}$$

for every ξ in $\mathbb{R}^N$, where $0 < \alpha \leq \beta$ are real numbers. Furthermore, M is symmetric.

Let us define the differential operator $\mathcal{L} : W_0^{1,2}(\Omega) \to W^{-1,2}(\Omega)$ by

$$\mathcal{L}(v) = -\mathrm{div}(M(x)\nabla v), \qquad v \in W_0^{1,2}(\Omega).$$

Thanks to the assumptions on M, $\mathcal{L}$ is linear, coercive, selfadjoint, and surjective.

In this paper, we are going to study the existence of solutions for a class of elliptic systems whose main example is the following:

$$\begin{cases} \mathcal{L}(\zeta) + \zeta - \mathrm{div}(\zeta\, A(x)\nabla u) = f & \text{in } \Omega, \\ \mathcal{L}(u) + u + \theta\, A(x)\nabla u \cdot \nabla u = \zeta^p & \text{in } \Omega, \\ \zeta = 0 = u & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

Here

$$p > 0, \quad 0 < \theta < 1,$$

and $f \geq 0$ is a function in some Lebesgue space.

Coupled systems similar to (1.3) appear, for example, in the theory of mean-field games introduced in [23–25]. In this context, even when the matrices A and M are smooth, and f is a bounded function, the existence of bounded solutions is not clear due to the growth of the coupling term ζ^p.

In the case of mean-field games systems, it is known from [16] that solutions are bounded, for any choice of the exponent p, if $A(x) = M(x)$ and f belongs to $L^\infty(\Omega)$; this result is proved through a change of variable which transforms the problem into a weakly coupled system of semilinear equations. We notice that the same proof of [16] would also work for problem (1.3) provided $\theta < 1$. However, for a general choice of $A(x)$ and $M(x)$, even possibly smooth, and a general growth p, the question of boundedness of solutions is still open, only partial results have been obtained so far. In particular, boundedness (and then smoothness, for smooth matrices) of solutions is known if the function ζ^p is replaced by a logarithm or if the growth exponent p does not exceed a certain value, see [19–22,25], and the most recent preprint [26] where the growth limitation for p is $p \leq \frac{2}{N}$. Further developments and estimates obtained with different methods, which especially apply to nonlinearities which are possibly decreasing with respect to ζ, appear in the recent preprint [17].