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Integration by parts and Pohozaev identities for space-dependent fractional-order operators

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Abstract

Consider a classical elliptic pseudodifferential operator P on $\mathbb{R}^n$ of order $2a$ $(0 < a < 1)$ with even symbol. For example, $P = A(x, D)^a$ where $A(x, D)$ is a second-order strongly elliptic differential operator; the fractional Laplacian $(-\Delta)^a$ is a particular case. For solutions u of the Dirichlet problem on a bounded smooth subset $\Omega \subset \mathbb{R}^n$, we show an integration-by-parts formula with a boundary integral involving $(d^{-a}u)|_{\partial\Omega}$, where $d(x) = \operatorname{dist}(x, \partial\Omega)$. This extends recent results of Ros-Oton, Serra and Valdinoci, to operators that are x-dependent, nonsymmetric, and have lower-order parts. We also generalize their formula of Pohozaev-type, that can be used to prove unique continuation properties, and nonexistence of nontrivial solutions of semilinear problems. An illustration is given with $P = (-\Delta + m^2)^a$. The basic step in our analysis is a factorization of P, $P \sim P^- P^+$, where we set up a calculus for the generalized pseudodifferential operators $P^\pm$ that come out of the construction.

Keywords: Fractional Laplacian; Nonlocal Dirichlet problem; Pohozaev identity; Pseudodifferential operator; a-Transmission property; Symbol factorization

1. Introduction

A prominent example of a fractional-order pseudodifferential operator (ψdo) is the fractional Laplacian $(-\Delta)^a$ on $\mathbb{R}^n$, $0 < a < 1$;

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$$(-\Delta)^a u = \operatorname{Op}(|\xi|^{2a})u = \mathcal{F}^{-1}(|\xi|^{2a}\hat{u}(\xi)), \quad \hat{u}(\xi) = \mathcal{F}u = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} u(x)\,dx. \tag{1.1}$$

It is currently of great interest in probability, finance, mathematical physics and differential geometry. It can also be described as a singular integral operator

$$(-\Delta)^a u(x) = c_{n,a} PV \int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2a}}\,dy = c_{n,a} PV \int_{\mathbb{R}^n} \frac{u(x)-u(x+y)}{|y|^{n+2a}}\,dy, \tag{1.2}$$

with convolution kernel $c_{n,a}|y|^{-n-2a} = \mathcal{F}^{-1}|\xi|^{2a}$.

Both descriptions allow generalizations. In (1.1), one can replace the symbol $|\xi|^{2a}$ by a more general nonvanishing function $p_0(x,\xi) \in C^\infty(\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}))$ that is homogeneous in ξ of degree $2a$, and add terms of lower order, to get a classical ψdo symbol $p(x,\xi)$; the operator is then no longer translation-invariant nor symmetric. Such operators are standard examples in the pseudodifferential calculus, and their boundary value problems on suitably smooth subsets Ω of $\mathbb{R}^n$ have been treated in works of Vishik and Eskin, cf. e.g. [10], Duduchava et al. [9,6], and Shargorodsky [39], with results on solvability in limited ranges of Sobolev spaces. Recently, a new boundary value theory has been presented in Grubb [16,17], obtaining regularity estimates of solutions divided by d^a ($d(x) = \operatorname{dist}(x, \partial\Omega)$) in full scales of function spaces with orders $s \to \infty$, for example in Hölder spaces of arbitrarily high order.

The pseudodifferential theory is useful in allowing a direct treatment of x-dependent operators, providing solution operators (or parametrices) that can give more efficient regularity estimates than the technique of perturbation of constant–coefficient cases.

In (1.2), one can replace the function $c_{n,a}|y|^{-n-2a}$ by other positive functions $K(y)$ that are homogeneous in y of degree $-n-2a$ and possibly less smooth. (In the smooth case this coincides with ψdo's with homogeneous x-independent symbol.) Such cases and further generalizations have recently been studied in probability and nonlinear analysis, see e.g. Caffarelli and Silvestre [8], Ros-Oton and Serra [29,33], and their references. For problems on bounded domains Ω, the integral operator methods allow limited smoothness of the integrand and boundary. To our knowledge, they have with few exceptions been applied to x-independent (translation-invariant) positive selfadjoint operators.

In the generalizations of (1.1) and (1.2), the fact that $|\xi|^{2a}$ is *even* (takes the same value at ξ and $-\xi$) is kept as a hypothesis, that p_0 is even in ξ, resp. that K is even in y.

The methods used in the pseudodifferential theory are complex, and differ radically from the real methods currently used for the singular integral formulations.

There is a large number of preceding studies of boundary problems for $(-\Delta)^a$ and its generalizations; let us mention e.g. [2,26,19,25,7,24,40,27,38,1,12,11,4].

A useful tool in solvability studies for linear and nonlinear partial differential equations on subsets $\Omega \subset \mathbb{R}^n$ is integration-by-parts formulas, Green's formulas. It is by no means obvious how one can establish such formulas for the present nonlocal operators. Interesting generalizations have recently been obtained for translation-invariant operators by Ros-Oton and Serra, partly with Valdinoci, in [30,34], and applied to nonlinear equations $Pu = f(u)$ there as well as in [31,32]; they have also been applied to nonlinear time-dependent Schrödinger equations by Boulenger, Himmelsbach and Lenzmann in [3].

In the present paper we show an extension of the formulas to x-dependent pseudodifferential operators, by completely different methods. The key ingredient is a factorization of P, $P \sim$