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Lifespan of solutions to the damped wave equation with a critical nonlinearity

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Abstract

In the present paper, we study a lifespan of solutions to the Cauchy problem for semilinear damped wave equations

$$\begin{cases} \partial_t^2 u - \Delta u + \partial_t u = f(u), & (t,x) \in [0, T(\varepsilon)) \times \mathbb{R}^n, \\ u(0,x) = \varepsilon u_0(x), & x \in \mathbb{R}^n, \\ \partial_t u(0,x) = \varepsilon u_1(x), & x \in \mathbb{R}^n, \end{cases} \tag{DW}$$

where $n \geq 1$, $f(u) = \pm|u|^{p-1}u$ or $|u|^p$, $p \geq 1$, $\varepsilon > 0$ is a small parameter, and (u_0, u_1) is a given initial data. The main purpose of this paper is to prove that if the nonlinear term is $f(u) = |u|^p$ and the nonlinear power is the Fujita critical exponent $p = p_F = 1 + \frac{2}{n}$, then the upper estimate to the lifespan is estimated by

$$T(\varepsilon) \leq \exp(C\varepsilon^{-p})$$

for all $\varepsilon \in (0, 1]$ and suitable data (u_0, u_1), without any restriction on the spatial dimension. Our proof is based on a test-function method utilized by Zhang [35]. We also prove a sharp lower estimate of the lifespan $T(\varepsilon)$ to (DW) in the critical case $p = p_F$.

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1. Introduction

In this paper, we consider the Cauchy problem for the semilinear damped wave equation over $\mathbb{R}^n$:

$$\begin{cases} \partial_t^2 u - \Delta u + \partial_t u = f(u), & (t,x) \in [0, T(\varepsilon)) \times \mathbb{R}^n, \\ u(0,x) = \varepsilon u_0(x), & x \in \mathbb{R}^n, \\ \partial_t u(0,x) = \varepsilon u_1(x), & x \in \mathbb{R}^n, \end{cases} \tag{1.1}$$

where $T > 0, n \in \mathbb{N}$ is the space dimension, $u = u(t,x)$ is a real valued unknown function of (t,x), $f(u) = \pm|u|^{p-1}u$ or $|u|^p$, $p \geq 1$, u_0 and u_1 are real valued initial data and $\varepsilon > 0$ is a small parameter.

Our main aim in the present paper is to derive an upper estimate of lifespan for solutions to (1.1) with $f(u) = |u|^p$ in the critical case, i.e. $p = p_F := 1 + 2/n$. More precisely, let $T(\varepsilon)$ be the maximal existence time of solutions to (1.1) with $f(u) = |u|^{p_F}$. We then derive that if the initial data $(u_0, u_1) \in H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ and it satisfies

$$u_0 + u_1 \in L^1(\mathbb{R}^n) \quad \text{and} \quad \int_{\mathbb{R}^n} (u_0(x) + u_1(x))dx > 0, \tag{1.2}$$

then $T(\varepsilon)$ satisfies

$$T(\varepsilon) \leq \exp(C\varepsilon^{-p}) \tag{1.3}$$

for all $\varepsilon \in (0, 1]$, where $C = C(n, u_0, u_1)$ is some positive constant independent of ε.

The sharp upper estimate had been obtained in the critical case $p = p_F$ in low dimensions, i.e. $n = 1, 2, 3$, by Li–Zhou [18] ($n = 1, 2$) and Nishihara [22] ($n = 3$). However, any upper estimate represented by ε has not been known in higher dimensional case $n \geq 4$.

We recall some previous results about (1.1). Local well-posedness in the energy space $H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ is well known in the case $1 \leq p \leq n/(n-2)$, $(n \geq 3)$ or $1 \leq p < \infty$, $(n = 1, 2)$ (see e.g. [28,4]). There are also many global well-posedness results for (1.1) (see [6,8,12,18,21,22,25,28,31,35,36] and their references therein). Among these papers, in the case of $f(u) = |u|^p$, Todorova and Yordanov [31] determined a critical exponent p_c for (1.1) as

$$p_c = p_F := 1 + 2/n.$$

Here the meaning of the critical exponent p_c stands for a number that classifies the global existence and non-existence for the small solution, namely if $p_c < p \leq p_L$, where p_L is the exponent for the local existence, then the initial value problem (1.1) has a small data global solution (SDGE), and if $1 < p < p_c$, the solution blows up in finite time even for small data. We note