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On the number of limit cycles for perturbed pendulum equations

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Abstract

We consider perturbed pendulum-like equations on the cylinder of the form $\ddot{x} + \sin(x) = \varepsilon \sum_{s=0}^{m} Q_{n,s}(x)\,\dot{x}^s$ where $Q_{n,s}$ are trigonometric polynomials of degree n, and study the number of limit cycles that bifurcate from the periodic orbits of the unperturbed case $\varepsilon = 0$ in terms of m and n. Our first result gives upper bounds on the number of zeros of its associated first order Melnikov function, in both the oscillatory and the rotary regions. These upper bounds are obtained expressing the corresponding Abelian integrals in terms of polynomials and the complete elliptic functions of first and second kind. Some further results give sharp bounds on the number of zeros of these integrals by identifying subfamilies which are shown to be Chebyshev systems.

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1. Introduction

The so-called *Hilbert's 16th Problem* was proposed by David Hilbert at the Paris conference of the International Congress of Mathematicians in 1900. The problem is to determine the upper

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bound for the number of limit cycles in two-dimensional polynomial vector fields of degree d, and to investigate their relative positions, see [11,15]. There is also a weaker version, the so-called *infinitesimal or tangential Hilbert's 16th Problem*, proposed by Arnold, which can be stated in the following way: let ω be a real 1-form with polynomial coefficients of degree at most d, and consider a polynomial H of degree $d+1$. A closed connected component of a level curve of $H = h$, denoted by γ_h, is called an *oval* of H. These ovals form continuous families. The infinitesimal Hilbert's 16th Problem then asks for an upper bound $V(d)$ of the number of real zeros of the Abelian integral

$$I(h) = \int_{\gamma_h} \omega.$$

The bound should be uniform with respect to the polynomial H, the family of ovals $\{\gamma_h\}$ and the form ω, i.e. it should only depend on the degree d, cf. [11,10]. The existence of $V(d)$ goes back to the works of Khovanskii and Varchenko [14,21]. Recently an explicit (non realistic) bound for $V(d)$ has been given in [2] by Binyamini, Novikov and Yakovenko.

There is a beautiful relationship between limit cycles and zeros of Abelian integrals: Consider a small deformation of a Hamiltonian vector field

$$X_\varepsilon = X_H + \varepsilon Y,$$

where $X_H = -H_y\partial_x + H_x\partial_y$, $Y = P\partial_x + Q\partial_y$ and $\varepsilon > 0$ is a small parameter. Denote by $d(h, \varepsilon)$ the *displacement function of the Poincaré map* of X_ε and consider its power series expansion in ε. The coefficients in this expansion are called *Melnikov functions* $M_k(h)$. Therefore, the limit cycles of the vector field correspond to isolated zeros of the first non-vanishing Melnikov function. A closed expression of the first Melnikov function $M_1(h) = I(h)$ was obtained by Pontryagin which is given by the Abelian integral

$$I(h) = \int_{\gamma_h} \omega, \quad \text{with} \quad \omega = P\,dy - Q\,dx.$$

Hence the number of isolated zeros of $I(h)$, counting multiplicity, provide an upper bound for the number of ovals of H that generate limit cycles of X_ε for ε close to zero. The coefficients of P and Q are considered as parameters, and so $I(h)$ splits into a linear combination $I(h) = \alpha_0 I_0(h) + \cdots + \alpha_\ell I_\ell(h)$, for some $\ell \in \mathbb{N}$, where the coefficients α_k depend on initial parameters and $I_k(h)$ are Abelian integrals with some $\omega_k = x^{i_k} y^{j_k} dx$. Therefore, the problem of finding the maximum number of isolated zeros of $I(h)$ is equivalent to finding an upper bound for the number of isolated zeros of any function belonging to the vector space generated by $I_j(h)$, $j = 0, \dots \ell$. This equivalent problem becomes easier when the basis of this vector space is a *Chebyshev system*, see Section 3 for details.

We are interested in these considerations because we want to analyze in terms of m and n the number of periodic orbits for perturbed pendulum-like equations of the form

$$\ddot{x} + \sin(x) = \varepsilon \sum_{s=0}^{m} Q_{n,s}(x)\,\dot{x}^s, \tag{1}$$