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Stability of the unique continuation for the wave operator via Tataru inequality and applications

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Abstract

In this paper we study the stability of the unique continuation in the case of the wave equation with variable coefficients independent of time. We prove a logarithmic estimate in an arbitrary domain of $\mathbb{R}^{n+1}$, where all the parameters are calculated explicitly in terms of the C^1-norm of the coefficients and on the other geometric properties of the problem. We use the Carleman-type estimate proved by Tataru in 1995 and an iteration of the local stability. We apply the result to the case of a wave equation with data on a cylinder and we get a stable estimate for any positive time, also after the first conjugate point associated with the geodesics of the metric of the variable coefficients.

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1. Introduction

We consider the wave operator in $\mathbb{R}^{n+1}$,

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$$P(y, D) = -D_0^2 + \sum_{j,k=1}^{n} g^{jk}(x) D_j D_k + \sum_{j=1}^{n} h^j(x) D_j + q(x), \tag{1.1}$$

where $y = (t, x) \in \mathbb{R} \times \mathbb{R}^n$ are the time–space variables, $D_0 = -i\partial_t$, $D_j = -i\partial_{x_j}$. The coefficients $g^{jk} \in C^1(\mathbb{R}^n)$ are real and independent of time, and $[g^{jk}]$ is a symmetric positive-definite matrix. The coefficients $h^j, q \in C^0(\mathbb{R}^n)$ are complex valued and independent of time.

An operator $P(y, D)$ is said to have the unique continuation property if for any solution u to $Pu = 0$ in a connected open set $\Omega \subset \mathbb{R}^{n+1}$ and vanishing on an open subset $B \subset \Omega$, it follows that u vanishes in Ω.

In the paper [21] Tataru proved for the first time the unique continuation property for (1.1) across every non-characteristic C^2-hypersurface with no limitation to the normal direction. The key point of these results is a Carleman-type estimate involving an exponential pseudo-differential operator.

Much is known about the consequences of the general unique continuation property for the corresponding Cauchy problem. Actually the unique continuation property has proved to be instructive in many areas of mathematics, e.g. in studying the uniqueness for linear and nonlinear PDEs together with their blow up or traveling wave solutions [8], in studying the Anderson localization [5], in control theory to get controllability results [23,24], in inverse problems to obtain uniqueness and stability estimates [13]. In particular Tataru's result [21] is crucial for the development of the Boundary Control method (see [3] for pioneering work and [12] for detailed exposition of the further developments).

Concerning the continuous dependence of the unique continuation property, that is its stability, less results are available. The elliptic and the parabolic cases have been studied in several settings by using either Carleman estimates or some versions of the three ball theorem (see [1], for a review of the results).

To our knowledge the hyperbolic case like (1.1) is still open for arbitrary domains and arbitrary matrix valued coefficients $g^{jk}(x)$, while there exist results for particular coefficients or domains (see [17,25]). This is maybe related to the difficulty of using the standard Carleman estimates for hyperbolic operators in order to prove the unique continuation close to the characteristic directions, that is the reason why Tataru's work was so important in this field.

The aim of the present work is then to prove a global stability estimate for the unique continuation of the operator $P(y, D)$.

In a previous work [4] we proved this property for the local case. Namely, given $S = \{y \in \Omega; \psi(y) = 0\}$ a $C^{2,\rho}$-smooth oriented hypersurface, which is non-characteristic in Ω, for some fixed $\rho \in (0, 1)$, we assume that $u \in H^1(\Omega)$ is supported in $\{y; \psi(y) \leq 0\} \cap \Omega$, and $P(y, D)u \in L^2(\Omega)$. Then, for each $y_0 \in S$, with $\psi'(y_0) \neq 0$, we find R, r with $R \geq 2r > 0$ such that the following stability estimate holds:

$$\|u\|_{L^2(B(y_0,r))} \leq c_{111} \frac{\|u\|_{H^1(B(y_0,2R))}}{\ln\left(1 + \frac{\|u\|_{H^1(B(y_0,2R))}}{\|Pu\|_{L^2(B(y_0,2R))}}\right)}.$$

Here $B(y_0, r)$ is a ball in $\mathbb{R}^{n+1}$ of radius $r > 0$ centered in y_0 and $B(y_0, r) \subset B(y_0, 2R) \subset \Omega$. The radii r and R and the coefficient c_{111} have been explicitly calculated with dependency on the geometric parameters and on the function ψ in [4].

In this work we use the previous local stability inequality to prove a similar logarithmic estimate for quite general domains of $\mathbb{R}^{n+1}$.