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Existence of stable solutions to $(-\Delta)^m u = e^u$ in $\mathbb{R}^N$ with $m \geq 3$ and $N > 2m$

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Abstract

We consider the polyharmonic equation $(-\Delta)^m u = e^u$ in $\mathbb{R}^N$ with $m \geq 3$ and $N > 2m$. We prove the existence of many entire stable solutions. This answers some questions raised by Farina and Ferrero in [7].

MSC: 35J91; 35B08; 35B35

Keywords: Polyharmonic equation; Entire stable solution

1. Introduction

In this paper, we are interested in the existence of entire stable solutions of the polyharmonic equation

$$(-\Delta)^m u = e^u \quad \text{in } \mathbb{R}^N. \tag{1.1}$$

with $m \geq 3$ and $N > 2m$.

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Definition 1. A solution u to (1.1) is said to be stable in $\Omega \subseteq \mathbb{R}^N$ if

$$\begin{cases} \int_\Omega |\nabla(\Delta^{\frac{m-1}{2}}\phi)|^2 dx - \int_\Omega e^u \phi^2 dx \geq 0 & \text{for any } \phi \in C_0^\infty(\Omega), \quad \text{when } m \text{ is odd;} \\ \int_\Omega |\Delta^{\frac{m}{2}}\phi|^2 dx - \int_\Omega e^u \phi^2 dx \geq 0 & \text{for any } \phi \in C_0^\infty(\Omega), \quad \text{when } m \text{ is even.} \end{cases}$$

Moreover, a solution to (1.1) is said to be stable outside a compact set K if it's stable in $\mathbb{R}^N \setminus K$. For simplicity, we say also that u is stable if $\Omega = \mathbb{R}^N$.

For $m = 1$, Farina [6] showed that (1.1) has no stable classical solution in $\mathbb{R}^N$ for $1 \leq N \leq 9$. He also proved that any classical solution which is stable outside a compact set in $\mathbb{R}^2$ verifies $e^u \in L^1(\mathbb{R}^2)$, therefore u is provided by the stereographic projection thanks to Chen–Li's classification result in [3], that is, there exist $\lambda > 0$ and $x_0 \in \mathbb{R}^2$ such that

$$u(x) = \ln\left[\frac{32\lambda^2}{(4 + \lambda^2|x - x_0|^2)^2}\right] \quad \text{for some } \lambda > 0. \tag{1.2}$$

Later on, Dancer and Farina [4] showed that (1.1) admits classical entire solutions which are stable outside a compact set of $\mathbb{R}^N$ if and only if $N \geq 10$.

It is well known that for any $m \geq 1$, $\lambda > 0$ and $x_0 \in \mathbb{R}^{2m}$, the function u defined in (1.2) resolves (1.1) in the conformal dimension $\mathbb{R}^{2m}$, there are the so-called spherical solutions, since they are provided by the stereographic projections.

For $m = 2$, the stability properties of entire solutions to (1.1) were studied in many works, especially the study for radial solutions is complete. Let $u(x) = u(r)$ be a smooth radial solution to (1.1), then u satisfies the following initial value problem

$$\begin{cases} (-\Delta)^m u = e^u, \\ u^{(2k+1)}(0) = 0, \quad \forall\, 0 \leq k \leq m-1, \\ \Delta^k u(0) = a_k, \quad \forall\, 0 \leq k \leq m-1. \end{cases} \tag{1.3}$$

Here the Laplacian Δ is seen as $\Delta u = r^{1-N}\left(r^{N-1}u'\right)'$ and a_k are constants in $\mathbb{R}$. Equivalently, let $v_k = (-\Delta)^k u$ for $0 \leq k \leq m-1$, the equation (1.3) can be written as a system

$$-v_k'' - \frac{N-1}{r}v_k' = v_{k+1} \text{ for } 0 \leq k \leq m-2; \quad \text{and } -v_{m-1}'' - \frac{N-1}{r}v_{m-1}' = e^{v_0} \tag{1.4}$$

where $v_k(0) = (-1)^k a_k$ and $v_k'(0) = 0$ for any $0 \leq k \leq m-1$.

Let $m = 2$, $a_0 = u(0) = 0$ (it's always possible by the scaling $u(\lambda x) + 2m\ln\lambda$). Denote by u_β the solution to (1.3) verifying $a_1 = \beta$, it's known from [1,5,11] that:

- There is no global solutions to (1.3) if $N \leq 2$.
- For $N \geq 3$, there exists $\beta_0 < 0$ depending on N such that the solution to (1.3) is globally defined, if and only if $\beta \leq \beta_0$.
- If $N = 3$ or 4, any entire solution u_β is unstable in $\mathbb{R}^N$, but stable outside a compact set.