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Uniqueness for an inverse problem in electromagnetism with partial data

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Abstract

A uniqueness result for the recovery of the electric and magnetic coefficients in the time-harmonic Maxwell equations from local boundary measurements is proven. No special geometrical condition is imposed on the *inaccessible* part of the boundary of the domain, apart from imposing that the boundary of the domain is $C^{1,1}$. The coefficients are assumed to coincide on a neighbourhood of the boundary, a natural property in applications.

Keywords: Inverse problem; Inverse boundary value problem; Maxwell equations; Partial data; Cauchy data set; Schrödinger equation

1. Introduction

Let μ, ε, σ be positive functions on a nonempty, bounded, open set Ω in $\mathbb{R}^3$, describing the permeability, permittivity and conductivity, respectively, of an inhomogeneous, isotropic medium Ω. Let $\partial\Omega$ denote the boundary of Ω and N the outward unit vector field normal to the boundary. Consider the electric and magnetic fields, E, H, satisfying the so-called time-harmonic Maxwell equations at a frequency $\omega > 0$, namely

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$$\begin{cases} \nabla \times H + i\omega\gamma E = 0, \\ \nabla \times E - i\omega\mu H = 0, \end{cases} \tag{1.1}$$

in Ω, where $\gamma = \varepsilon + i\sigma/\omega$, i denotes the imaginary unit, and $\nabla\times$ denotes the *curl* operator.

Let ε, σ, μ be non-negative coefficients and assume that ε, μ are bounded from below in Ω. Then there exist positive values of ω for which the equations (1.1), posed in proper spaces and domains, with the tangential boundary condition either $N \times H|_{\partial\Omega} = 0$ or $N \times E|_{\partial\Omega} = 0$, have non-trivial solutions (see [60,42]). Such values of ω are called *resonant frequencies*.

The boundary data corresponding to the inverse boundary value problem (IBVP) for the system (1.1) only can be given by a boundary mapping (the *impedance* or *admittance* map) if ω is not a resonant frequency. The fact that the position of resonant frequencies depends on the unknown coefficients (as it is stated in [52]) motivated Pedro Caro in [16] to consider a *Cauchy data set* instead of a boundary map as boundary data. Cauchy data sets have been used in [14,55,56,16,17,20].

This work is focused on the IBVP for the system (1.1) with local boundary measurements established by a Cauchy data set taken just on a part of $\partial\Omega$. More precisely, Definition 1.1 describes the conditions for the domain and the part of its boundary where the measurements are taken and Definition 1.2 (used in [17]) introduces the boundary data for the IBVP studied in this article.

Definition 1.1. Let $\Omega \subset \mathbb{R}^3$ be a non-empty, bounded domain in $\mathbb{R}^3$ with $C^{1,1}$ boundary $\partial\Omega$. Assume Γ is a smooth proper non-empty open subset of $\partial\Omega$. We call Γ the *accessible part* of the boundary $\partial\Omega$ and $\Gamma_c := \partial\Omega \setminus \overline{\Gamma}$ the *inaccessible part* of the boundary.

Definition 1.2. Let Ω and Γ be as in Definition 1.1. For a pair of smooth coefficients μ, γ on Ω according to Definition 1.4, define the Cauchy data set restricted to Γ, write $C(\mu, \gamma; \Gamma)$, at frequency $\omega > 0$ by the set of couples $(T, S) \in TH_0(\Gamma) \times TH(\Gamma)$ such that there exists a solution $(E, H) \in (H(\Omega; \mathrm{curl}))^2$ of (1.1) in Ω satisfying $N \times E|_{\partial\Omega} = T$ and $N \times H|_{\Gamma} = S$, where the spaces $H(\Omega; \mathrm{curl})$, $TH(\Gamma)$, $TH_0(\Gamma)$ are defined in Definition 1.3.

It is known that if the domain Ω is not convex and its boundary is not $C^{1,1}$, Maxwell equations may not admit solutions in $H^1(\Omega)$ even for boundary data in $H^{1/2}(\partial\Omega)$ (see [8,9,57,58,23]). Thus, for a less regular domain (e.g., Lipschitz), some non-standard Sobolev spaces are necessary. Some of them, which will be used in these notes, appear in the following

Definition 1.3. Let Ω and Γ be as in Definition 1.1. Define $H^{1/2}(\Gamma) = \{f|_\Gamma : f \in H^{1/2}(\partial\Omega)\}$, with norm $\|g\|_{H^{1/2}(\Gamma)} = \inf\{\|f\|_{H^{1/2}(\partial\Omega)} : f|_\Gamma = g\}$, and $H_0^{1/2}(\Gamma) = \{f \in H^{1/2}(\partial\Omega) : \mathrm{supp}\, f \subset \overline{\Gamma}\}$, with norm $\|f\|_{H_0^{1/2}(\Gamma)} = \|f\|_{H^{1/2}(\partial\Omega)}$. Write $H^{-1/2}(\partial\Omega)$ for the dual space of $H^{1/2}(\partial\Omega)$. Consider the space $H(\Omega; \mathrm{curl}) = \{u \in L^2(\Omega; \mathbb{C}^3) : \nabla \times u \in L^2(\Omega; \mathbb{C}^3)\}$ with the usual graph norm, and the following

$$TH(\partial\Omega) = \{u \in H^{-1/2}(\partial\Omega; \mathbb{C}^3) : N \times v|_{\partial\Omega} = u, \text{ for some } v \in H(\Omega; \mathrm{curl})\},$$

$$TH(\Gamma) = \{u|_\Gamma : u \in TH(\partial\Omega)\},$$

$$TH_0(\Gamma) = \{u \in TH(\partial\Omega) : \mathrm{supp}\, u \subset \overline{\Gamma}\},$$