Available online at www.sciencedirect.com

ScienceDirect

J. Differential Equations 260 (2016) 6604–6637

Journal of Differential Equations

www.elsevier.com/locate/jde

On the well-posedness of 2-D incompressible Navier–Stokes equations with variable viscosity in critical spaces ☆

Huan Xu [a,b,*], Yongsheng Li [a], Xiaoping Zhai [a]

[a] *School of Mathematics, South China University of Technology, Guangzhou, Guangdong 510640, PR China*

[b] *Department of Mathematics, Auburn University, Auburn, AL 36849, USA*

Received 14 October 2015

Available online 21 January 2016

Abstract

In this paper, we first prove the local well-posedness of the 2-D incompressible Navier–Stokes equations with variable viscosity in critical Besov spaces with negative regularity indices, without smallness assumption on the variation of the density. The key is to prove for $p \in (1, 4)$ and $a \in \dot{B}^{\frac{2}{p}}_{p,1}(\mathbb{R}^2)$ that the solution mapping $\mathcal{H}_a : F \mapsto \nabla\Pi$ to the 2-D elliptic equation $\operatorname{div}\big((1+a)\nabla\Pi\big) = \operatorname{div} F$ is bounded on $\dot{B}^{\frac{2}{p}-1}_{p,1}(\mathbb{R}^2)$. More precisely, we prove that

$$\|\nabla\Pi\|_{\dot{B}^{\frac{2}{p}-1}_{p,1}} \le C\big(1 + \|a\|_{\dot{B}^{\frac{2}{p}}_{p,1}}\big)^2 \|F\|_{\dot{B}^{\frac{2}{p}-1}_{p,1}}.$$

The proof of the uniqueness of solution to (1.2) relies on a Lagrangian approach [15–17]. When the viscosity coefficient $\mu(\rho)$ is a positive constant, we prove that (1.2) is globally well-posed.

MSC: 35Q35; 76D03; 35B45

☆ This work is supported by NSFC under grant numbers 11171116 and 11571118, and by the Fundamental Research Funds for the Central Universities of China under the grant number 2014ZG0031.

* Corresponding author at: School of Mathematics, South China University of Technology, Guangzhou, Guangdong 510640, PR China.

E-mail addresses: hxuscut@163.com (H. Xu), yshli@scut.edu.cn (Y.S. Li), pingxiaozhai@163.com (X.P. Zhai).

http://dx.doi.org/10.1016/j.jde.2016.01.007

0022-0396/

Keywords: Incompressible Navier–Stokes equations; Littlewood–Paley theory; Lagrangian coordinates; Well-posedness

1. Introduction

In this paper, we study the Cauchy problem of the 2-D incompressible Navier–Stokes equations with variable viscosity in critical Besov spaces

$$\begin{cases} \partial_t \rho + u \cdot \nabla \rho = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - \operatorname{div}\big(2\mu(\rho)\mathcal{M}(u)\big) + \nabla \Pi = 0, \\ \operatorname{div} u = 0, \\ (\rho, u)|_{t=0} = (\rho_0, u_0), \end{cases} \tag{1.1}$$

where ρ and $u = (u_1, u_2)$ stand for the density and velocity field, $\mathcal{M}(u) = \frac{1}{2}(\partial_i u_j + \partial_j u_i)$, Π is a scalar pressure function, the viscosity coefficient $\mu(\rho)$ is smooth, positive on $[0, \infty)$. Throughout, we assume that the space variable x belongs to the whole space $\mathbb{R}^2$.

Global weak solutions with finite energy to system (1.1) were first obtained by the Russian school [6] in the case when $\mu(\rho) = \mu > 0$ and ρ_0 is bounded away from 0. We also refer to [24] for an overview of results on weak solutions and to [18–20] for some improvements. However, the uniqueness of weak solutions is not known in general. When $\mu(\rho) = \mu > 0$ and ρ_0 is bounded away from 0, Ladyzhenskaya and Solonnikov [23] initiated the studies for unique solvability of system (1.1) in a bounded domain Ω with homogeneous Dirichlet boundary condition for u. Similar results were established by Danchin [11] in the whole space $\mathbb{R}^n$ with initial data in the almost critical Sobolev spaces. On the other hand, from the viewpoint of physics, it is interesting to study the case for which density is discontinuous. Recently, Danchin and Mucha [17] proved by using a Lagrangian approach that the system (1.1) has a unique local solution with initial data $(\rho_0, u_0) \in L^\infty(\mathbb{R}^n) \times H^2(\mathbb{R}^n)$ if initial vacuum dose not occur, see also some improvements and generalizations in [21,22,25].

On the other hand, if the density ρ is away from zero, we denote by $a \stackrel{\text{def}}{=} \frac{1}{\rho} - 1$ and $\tilde{\mu}(a) \stackrel{\text{def}}{=} \mu(\frac{1}{1+a})$ so that the system (1.1) can be equivalently reformulated as

$$\begin{cases} \partial_t a + u \cdot \nabla a = 0, \\ \partial_t u + u \cdot \nabla u - (1 + a)\left\{\operatorname{div}\big(2\tilde{\mu}(a)\mathcal{M}(u)\big) - \nabla \Pi\right\} = 0, \\ \operatorname{div} u = 0, \\ (a, u)|_{t=0} = (a_0, u_0). \end{cases} \tag{1.2}$$

Just as the classical Navier–Stokes equations, the system (1.2) also has a scaling. Indeed, if (a, u) solves (1.2) with initial data (a_0, u_0), then for any $\lambda > 0$,

$$(a, u)_\lambda(t, x) \stackrel{\text{def}}{=} (a(\lambda^2 t, \lambda x), \lambda u(\lambda^2 t, \lambda x))$$

also solves (1.2) with initial data $(a_0(\lambda\cdot), \lambda u_0(\lambda\cdot))$. Moreover, the norm of $(a_0(\lambda\cdot), \lambda u_0(\lambda\cdot))$ is independent of λ in the so-called critical spaces $\dot{B}^{\frac{2}{p}}_{p,1}(\mathbb{R}^2) \times \dot{B}^{\frac{2}{p}-1}_{p,1}(\mathbb{R}^2)$. In resent ten years,