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One-dimensional solutions of non-local Allen–Cahn-type equations with rough kernels

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Abstract

We are interested in the study of local and global minimizers for an energy functional of the type

$$\frac{1}{4} \iint\limits_{\mathbb{R}^{2N} \setminus (\mathbb{R}^N \setminus \Omega)^2} |u(x) - u(y)|^2 K(x-y)\, dx\,dy + \int\limits_{\Omega} W(u(x))\, dx,$$

where W is a smooth, even double-well potential and K is a non-negative symmetric kernel in a general class, which contains as a particular case the choice $K(z) = |z|^{-N-2s}$, with $s \in (0,1)$, related to the fractional Laplacian. We show the existence and uniqueness (up to translations) of one-dimensional minimizers in the full space $\mathbb{R}^N$ and obtain sharp estimates for some quantities associated to it. In particular, we deduce the existence of solutions of the non-local Allen–Cahn equation

$$\text{p.v.} \int\limits_{\mathbb{R}^N} (u(x) - u(y))\, K(x-y)\, dy + W'(u(x)) = 0 \quad \text{for any } x \in \mathbb{R}^N,$$

which possess one-dimensional symmetry.

The results presented here were proved in [10,36,9] for the model case $K(z) = |z|^{-N-2s}$. In our work, we consider instead general kernels which may be possibly non-homogeneous and truncated at infinity.

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Contents

1. Introduction and description of the model

In the present paper we are concerned with a minimization problem related to phase transition phenomena. We study monotone entire minimal configurations for a total energy functional obtained by coupling a standard Gibbs-type free energy with a non-local penalization term modeled upon a Gagliardo-type seminorm. The novelty of our work mostly resides in the introduction of this last term, thanks to which we are able to encompass the presence of long-range interactions between the particles constituting the medium. In particular, our model is general enough to allow for anisotropic effects (possibly changing at different scales of distances, too) and both finite- and infinite-range interactions.

We now proceed to the formal description of the setting.

Given a domain $\Omega \subseteq \mathbb{R}^N$, for some integer $N \geq 1$, we consider the energy functional

$$\mathscr{E}_K(u, \Omega) := \mathscr{K}_K(u, \Omega) + \mathscr{P}(u, \Omega), \tag{1.1}$$