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J. Differential Equations 260 (2016) 6697–6715

Journal of Differential Equations

www.elsevier.com/locate/jde

The entry–exit function and geometric singular perturbation theory

Peter De Maesschalck [a], Stephen Schecter [b,*]

[a] *Department of Mathematics and Statistics, Hasselt University, B-3590 Diepenbeek, Belgium*
[b] *Department of Mathematics, North Carolina State University, Raleigh, NC 27695-8205, USA*

Received 22 October 2015; revised 18 December 2015

Available online 26 January 2016

Abstract

For small $\varepsilon > 0$, the system $\dot{x} = \varepsilon$, $\dot{z} = h(x, z, \varepsilon)z$, with $h(x, 0, 0) < 0$ for $x < 0$ and $h(x, 0, 0) > 0$ for $x > 0$, admits solutions that approach the x-axis while $x < 0$ and are repelled from it when $x > 0$. The limiting attraction and repulsion points are given by the well-known entry–exit function. For $h(x, z, \varepsilon)z$ replaced by $h(x, z, \varepsilon)z^2$, we explain this phenomenon using geometric singular perturbation theory. We also show that the linear case can be reduced to the quadratic case, and we discuss the smoothness of the return map to the line $z = z_0$, $z_0 > 0$, in the limit $\varepsilon \to 0$.

Keywords: Entry–exit function; Geometric singular perturbation theory; Bifurcation delay; Blow-up; Turning point

1. Introduction

Consider the slow–fast planar system

$$\dot{x} = \varepsilon f(x, z, \varepsilon), \tag{1.1}$$

$$\dot{z} = g(x, z, \varepsilon)z, \tag{1.2}$$

* Corresponding author.
E-mail addresses: peter.demaesschalck@uhasselt.be (P. De Maesschalck), schecter@math.ncsu.edu (S. Schecter).

http://dx.doi.org/10.1016/j.jde.2016.01.008

with $x \in \mathbb{R}$, $z \in \mathbb{R}$,

$$f(x,0,0) > 0; \quad g(x,0,0) < 0 \text{ for } x < 0 \text{ and } g(x,0,0) > 0 \text{ for } x > 0. \tag{1.3}$$

For $\varepsilon = 0$, the x-axis consists of equilibria; see Fig. 1.2(a) below. These equilibria are normally attracting for $x < 0$ and normally repelling for $x > 0$. For $\varepsilon > 0$, the x-axis remains invariant, and the flow on it is to the right. For small $\varepsilon > 0$, a solution that starts at (x_0, z_0), with x_0 negative and $z_0 > 0$ small, is attracted quickly toward the x-axis, then drifts to the right along the x-axis, and finally is repelled from the x-axis. It reintersects the line $z = z_0$ at a point whose x-coordinate we denote by $p_\varepsilon(x_0)$. As $\varepsilon \to 0$, the return map $p_\varepsilon(x_0)$ approaches a function $p_0(x_0)$ given implicitly by the formula

$$\int_{x_0}^{p_0(x_0)} \frac{g(x,0,0)}{f(x,0,0)}\, dx = 0. \tag{1.4}$$

In other words, the solution does not leave the x-axis as soon as it becomes unstable at $x = 0$; instead the solution stays near the x-axis until a repulsion has built up to balance the attraction that occurred before $x = 0$. The function p_0 is called the entry–exit [1] or way in–way out [5] function.

This phenomenon, in which a solution of a slow–fast system stays near a curve of equilibria of the slow limit system after it has become unstable, and leaves at a point given by an integral like (1.4), has been called "Pontryagin delay" [12] or "bifurcation delay" [2]. As far as we know, it was originally discovered in a different context, in which the fast variable z in (1.2) is two-dimensional and, for $\varepsilon = 0$, the equilibrium at $z = 0$ undergoes a Hopf bifurcation as x passes 0; see [15], which was written under the direction of Pontryagin. In this situation, it turns out that the delay phenomenon need not occur if the system is not analytic. See [13] for a recent survey.

For the system (1.1)–(1.3), Pontryagin delay and the entry–exit function are discussed in [12, 9,14,4]. Methods include asymptotic expansions [12,9], comparison to solutions constructed by separation of variables [14], and direct estimation of the solution and its derivatives using the variational equation [4]. The last paper gives the most complete results.

Note that for the system (1.1)–(1.3) with $\varepsilon = 0$, the line of equilibria along the x-axis loses normal hyperbolicity at the "turning point" $x = 0$. The blow-up method of geometric singular perturbation theory [7,10] is today the method of choice for understanding loss of normal hyperbolicity. However, unless nongenericity conditions are imposed at the turning point [6], neither spherical blow-up of the turning point nor cylindrical blow-up along the x-axis appears to help with this problem. Even in the nongeneric cases where blow-up does help, it probably does not yield optimal smoothness results.

Pontryagin delay is also encountered in the codimension-one bifurcation of slow–fast systems that gives rise to the solutions known as canards; see [1,5]. Consider for example the system

$$\dot{x} = \varepsilon f(x,z) = \varepsilon(ax + bz + \ldots), \tag{1.5}$$

$$\dot{z} = g(x,z) = -(x + cz^2 + \ldots), \tag{1.6}$$

with b and c positive. The omitted terms in the first equation are higher-order; those in the second consist of other quadratic terms and higher-order terms. This system is codimension-one in the