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Global well-posedness of the 2D Boussinesq equations with fractional Laplacian dissipation

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Abstract

As a continuation of the previous work [48], in this paper we focus on the Cauchy problem of the two-dimensional (2D) incompressible Boussinesq equations with fractional Laplacian dissipation. We give an elementary proof of the global regularity of the smooth solutions of the 2D Boussinesq equations with a new range of fractional powers of the Laplacian. The argument is based on the nonlinear lower bounds for the fractional Laplacian established in [13]. Consequently, this result significantly improves the recent works [13,45,48].

MSC: 35Q35; 35B65; 76D03

Keywords: 2D Boussinesq equations; Fractional Laplacian dissipation; Global regularity

1. Introduction

In this paper, we are interested in studying the following 2D incompressible Boussinesq equations with fractional Laplacian dissipation

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$$\begin{cases} \partial_t u + (u \cdot \nabla)u + \nu\Lambda^\alpha u + \nabla p = \theta e_2, & x \in \mathbb{R}^2,\ t > 0, \\ \partial_t \theta + (u \cdot \nabla)\theta + \kappa\Lambda^\beta \theta = 0, & x \in \mathbb{R}^2,\ t > 0, \\ \nabla \cdot u = 0, & x \in \mathbb{R}^2,\ t > 0, \\ u(x,0) = u_0(x), \quad \theta(x,0) = \theta_0(x), & x \in \mathbb{R}^2, \end{cases} \tag{1.1}$$

where the numbers $\nu \geq 0$, $\kappa \geq 0$, $\alpha \in [0,2]$ and $\beta \in [0,2]$ are real parameters. Here $u(x,t) = (u_1(x,t), u_2(x,t))$ is a vector field denoting the velocity, $\theta = \theta(x,t)$ is a scalar function denoting the temperature, p is the scalar pressure and $e_2 = (0,1)$. The fractional Laplacian operator Λ^α, $\Lambda := (-\Delta)^{\frac{1}{2}}$ denotes the Zygmund operator which is defined through the Fourier transform, namely

$$\widehat{\Lambda^\alpha f}(\xi) = |\xi|^\alpha \hat{f}(\xi),$$

where

$$\hat{f}(\xi) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{-ix\cdot\xi} f(x)\, dx.$$

The fractional dissipation operator serves to model many physical phenomena (see [17]) in hydrodynamics and molecular biology such as anomalous diffusion in semiconductor growth (see [36]). We remark the convention that by $\alpha = 0$ we mean that there is no dissipation in $(1.1)_1$, and similarly $\beta = 0$ represents that there is no dissipation in $(1.1)_2$.

The standard Boussinesq equations (namely $\alpha = \beta = 2$) are of relevance to study a number of models coming from atmospheric or oceanographic turbulence where rotation and stratification play an important role (see for example [32,35]). Moreover, as pointed out in [32], the 2D inviscid Boussinesq equations, namely (1.1) with $\alpha = \beta = 0$, are identical to the incompressible axi-symmetric (away from the z-axis) swirling 3D Euler equations. There are geophysical circumstances in which the Boussinesq equations with fractional Laplacian may arise. The effect of kinematic and thermal diffusion is attenuated by the thinning of atmosphere. This anomalous attenuation can be modeled by using the space fractional Laplacian (see [9,18]).

The global well-posedness of the 2D Boussinesq equations has recently drawn a lot of attention and many important results have been established. It is well-known that the system (1.1) with full Laplacian dissipation (namely, $\alpha = \beta = 2$) is global well-posed, see, e.g., [7]. In the case of inviscid Boussinesq equations, the global regularity problem turns out to be extremely difficult and remains outstandingly open. Therefore, it is natural to consider the intermediate cases. Actually, many important progresses have recently been made on this direction. Almost at the same time, Chae [10] and Hou and Li [23] proved the global regularity for the system (1.1) when $\alpha = 2$ and $\beta = 0$ or $\alpha = 0$ and $\beta = 2$ independently. Since then, much effort is devoted to the global regularity of (1.1) with the smallest possible $\alpha \in (0,2)$ and $\beta \in (0,2)$. As pointed out in [24], we can classify α and β into three categories: the subcritical case when $\alpha + \beta > 1$, the critical case when $\alpha + \beta = 1$ and the supercritical case when $\alpha + \beta < 1$. As a rule of thumb, with current methods it seems impossible to obtain the global regularity for the 2D Boussinesq equations with supercritical dissipation. Recently, Jiu, Wu and Yang [25] established the eventual regularity of weak solutions of the system (1.1) when α and β are in the suitable supercritical range. For the critical case, there are several works that are available. In the two elegant papers, Hmidi, Keraani and Rousset [21,22] established the global well-posedness result