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# The cohomological span of $\mathcal{LS}$-Conley index

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**Abstract**

In this paper we introduce a new homotopy invariant – the cohomological span of $\mathcal{LS}$-Conley index. We prove the theorems on the existence of critical points for a class of strongly indefinite functionals with the gradient of the form $Lx + K(x)$, where $L$ is bounded linear and $K$ is completely continuous. We give examples of Hamiltonian systems for which our methods give better results than the Morse inequalities. We also give a formula for the $\mathcal{LS}$-index of an isolated critical point, which is an extension of the classical Dancer theorem for the case of $\mathcal{LS}$-index.


*MSC:* 37B30; 58E05

*Keywords:* Conley index in Hilbert spaces; Strongly indefinite operator; Critical point theory

## 1. Introduction

The purpose of this paper is to present a new homotopy invariant, the cohomological span of $\mathcal{LS}$-Conley index, and to show how this invariant can be applied to obtain existence and multiplicity results for strongly indefinite functionals.

The existence problem for solutions of differential equation can be restated in the terms of existence problems for critical points of functionals defined on Hilbert spaces. This approach leads to consider strongly indefinite functionals, i.e. both stable and unstable manifolds at a

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critical point are of infinite dimension. There are many methods developed to deal with such a functionals including both variational and topological methods (e.g. [1–7]).

In this paper we are going to work with $\mathcal{LS}$-Conley index – the extension of classical Conley index for Hilbert spaces presented by K. Gęba, M. Izydorek and A. Pruszko in [8]. Further development of this theory was given by Izydorek in [9]. He defined the cohomology groups of $\mathcal{LS}$-index and gave the examples how this theory can be applied for the existence problems for periodic solution of Hamiltonian systems. Applications to PDE were given later by M. Izydorek and K. Rybakowski in [10].

In 1984, E. Dancer proved in [11] that if $p$ is a critical point of $F:\mathbb{R}^n \to \mathbb{R}$ then the classical Conley index of $\{p\}$ has the homotopy type of a k-fold suspension of some space connected with null space of $\nabla^2 F(p)$. In this paper we prove a similar statement for $\mathcal{LS}$-index (Theorem 3.8). This is a crucial observation for further results when we deal with cohomological span.

Let $X$ be an isolating neighborhood. The cohomological span of $\mathcal{LS}$-index $h_{\mathcal{LS}}(X)$ is defined to be a pair

$$\Gamma(h_{\mathcal{LS}}(X)) = (\underline{\gamma}(h_{\mathcal{LS}}(X)), \overline{\gamma}(h_{\mathcal{LS}}(X))),$$

where $\underline{\gamma}(h_{\mathcal{LS}}(X))$ and $\overline{\gamma}(h_{\mathcal{LS}}(X))$ stand for the numbers of the first and the last nontrivial cohomology groups of $\mathcal{LS}$-index $h_{\mathcal{LS}}(X)$. In some cases, if $\mathrm{Inv}X$ is an isolated critical point then the difference $|\Gamma(X)| = \overline{\gamma}(h_{\mathcal{LS}}(X)) - \underline{\gamma}(h_{\mathcal{LS}}(X))$ can be globally bounded by some constant $M$. Hence, if for some set $X$ we have $|\Gamma(X)| \geq M$ then $X$ does not contain only critical point (see Corollary 4.8). This observation, combined with examining certain long exact sequences, gives us the existence results. Given examples show that in some situations our theory allows to find more critical points than another widely-applied method, i.e. examining Morse inequalities.

To be more precise, assume that $S$ is an isolated invariant set and $S_1 \subset S$ is an isolated critical point. Suppose that Morse polynomials of $S_1$ and $S$ are $P(t, S_1) = t^0 + t^1$ and $P(t, S) = t^m$, $m \geq M$. In this case, the Morse equation

$$t^0 + t^1 = t^m + (1+t)Q(t)$$

is not satisfied. Therefore, there exists at least one critical point in $S$, say $p$, different to $S_1$. Letting $P(t, \{p\}) = t^m$ and $Q(t) = t^0$ we obtain equality $t^0 + t^1 + t^m = t^m + (1+t)Q(t)$, which is true for all $t$.

On the other hand we have $|\Gamma(S_1)| = 2$, $|\Gamma(S)| = 1$, $\underline{\gamma}(S) - \overline{\gamma}(S_1) = m - 1$ and

$$|\Gamma(S)| + |\Gamma(S_1)| + \underline{\gamma}(S) - \overline{\gamma}(S_1) - 2 = m \geq M.$$

Theorem 4.16 follows that $S \setminus S_1$ contains at least two critical points. This result cannot be obtained by examining Morse equation (cf. Section 5).

This paper is organized as follows. In Section 2 we recall basic definitions and facts about the $\mathcal{LS}$-Conley index theory. We refer the reader to [8,9] for more details. In the next section we prove the formula for $\mathcal{LS}$-index of an isolated critical point. In Section 4 we introduce the cohomological span and next we prove the theorems concerning the existence of critical points. Finally, in Section 5 we give simple examples to show how our theory works when asymptotically linear Hamiltonian systems are considered.