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A pseudo-extractor approach to hidden boundary regularity for the wave equation with mixed boundary conditions

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Abstract

In this paper we introduce a new approach to "hidden" boundary regularity for the linear wave equation with mixed Dirichlet–Neumann boundary conditions, where the Neumann data is non-smooth. First, we obtain existence and uniqueness of solution by Galerkin estimates. Then we use a new, pseudo-extractor technique (based on the Fourier transform and shape and tangential calculus) in order to provide sharp regularity for the solution at the boundary.

Keywords: Wave equation; Neumann boundary conditions; Pseudo-extractor; Boundary regularity

1. Introduction

In this paper we provide existence and uniqueness of solution, as well as hidden boundary regularity for the following wave equation with Dirichlet–Neumann boundary condition: Let $D \subset \mathbb{R}^n$ be fixed (potentially included in a C^2 manifold). Let $\Omega \subset D$ be an open bounded

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domain, with smooth boundary $\partial\Omega = \Gamma \cup S$, where $\text{meas}(S) \neq 0$ and $\bar{\Gamma} \cap \bar{S} = \emptyset$. We consider the following wave equation on $I = [0,\ 2\tau]$:

$$\begin{cases} y_{tt} - \Delta y = f & Q = \Omega \times I \\ \frac{\partial y}{\partial n} = g & \Sigma = \Gamma \times I \\ y = 0 & S \times I \\ y(0, x) = y_t(0, x) = 0 \end{cases} \tag{1.1}$$

Assumption 1.1. We impose the following conditions on the interior and boundary data f and g:

1. $f(0, x) = 0$ and $[f \in L^2(I \times \Omega)$ or $f \in W^{1,1}(I, (H^1_*(\Omega))')]$,
2. $g(0, x) = 0$ and $g \in W^{1,1}(I, H^{-1/2}(\Gamma))$.

The motivation and need for this analysis came from the investigation of strong shape differentiability for the wave equation with mixed (Dirichlet–Neumann) boundary conditions on a moving domain, where the Neumann data is non-smooth [2]. This is a fundamental question in shape optimization [21] and control problems for the linear wave equation and coupled systems where the hyperbolic equation is coupled with other dynamics, and the matching conditions at the boundary are of Neumann type [6]. Shape derivative analysis is now well known for various classical linear and nonlinear boundary value problems. However, the hyperbolic case is more delicate, precisely due to interior and boundary regularity of the solution. The case of Dirichlet boundary conditions has been solved in [5], where the authors took advantage of the well-known results provided in [13]. In comparison, Neumann boundary data that is non-zero, and non-smooth, induces the failure of the Lopatinski condition. It is well known in the literature that even for $L^2(\Sigma)$-Neumann boundary data, the maximal amount of regularity that one obtains for the solution to the linear wave equation is $C(0, T; H^{2/3}(\Omega))$ (solution does not reach finite-energy level) [13,17]. One way to overcome this is to introduce damping in the equation, which will then counteract the source and provide the desired finite energy regularity for the solution to the wave equation. It was observed in the literature that even linear dissipation acting in the interior of the domain (i.e. $g(u_t) = u_t$) changes the problem to one where the Lopatinski condition is satisfied.

In the shape differentiability analysis mentioned above [2], the undamped wave has Neumann boundary source $g \in W^{1,1}(H^{-1/2}(\Gamma))$. Implicitly, one does not a-priori have “good” interior and boundary regularity for the wave solution, which is a key ingredient in the differentiability analysis that one needs to perform.

The technique that we employ in this paper is a combination of Galerkin approximations with a careful adaptation of the “extractor technique”, which we call “pseudo-extractor technique” (described in Section 4). This is a different approach from [13], where the authors used pseudo-differential calculus, and functional analysis techniques based on cosine/sine operators.

The extractor technique was first introduced in [9]. The main idea behind it is to use shape calculus, and take advantage of the Lagrangian (material) and the Eulerian (shape) expressions for the shape derivative of a well chosen “energy-like” functional with respect to an autonomous Lipschitzian vector field in $\mathbb{R}^n$. This neat strategy was initially used to provide a fundamental identity as the foundation of the multiplier method for controllability problems [10].

The extractor technique was applied successfully for elliptic problems [9], and also for the wave equation with Dirichlet boundary conditions [5,20]. However, the extractor technique fails