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Study of a family of higher order nonlocal degenerate parabolic equations: From the porous medium equation to the thin film equation

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Abstract

In this paper, we study a nonlocal degenerate parabolic equation of order $\alpha + 2$ for $\alpha \in (0, 2)$. The equation is a generalization of the one arising in the modeling of hydraulic fractures studied by Imbert and Mellet in 2011. Using the same approach, we prove the existence of solutions for this equation for $0 < \alpha < 2$ and for nonnegative initial data satisfying appropriate assumptions. The main difference is the compactness results due to different Sobolev embeddings. Furthermore, for $\alpha > 1$, we construct a nonnegative solution for nonnegative initial data under weaker assumptions.

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1. Introduction

In this paper, we study the following problem:

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$$\begin{cases} \partial_t u + \partial_x(u^n \partial_x I(u)) = 0 & \text{for } x \in \Omega, \quad t > 0, \\ \partial_x u = 0, u^n \partial_x I(u) = 0 & \text{for } x \in \partial\Omega, \quad t > 0, \\ u(0,x) = u_0(x) & \text{for } x \in \Omega, \end{cases} \tag{1}$$

where $\Omega = (a,b)$ is a bounded interval in $\mathbb{R}$, n is a positive real number and I is a nonlocal elliptic negative operator of order α defined as the $\alpha/2$ power of the Laplace operator with Neumann boundary conditions $I = -(-\Delta)^{\frac{\alpha}{2}}$ where $\alpha \in (0,2)$; this operator will be defined below by using the spectral decomposition of the Laplacian.

The case $\alpha = 1$ was studied by Imbert and Mellet [15] who proved the existence of nonnegative solutions for nonnegative initial data with appropriate conditions. In this case, when $n = 3$ the equation designs the physical KGD model developed by Geertsma and de Klerk [9] and Khristianovich and Zheltov [21]. It represents the influence of the pressure exerted by a viscous fluid on a fracture in an elastic medium subject only to plane strain. This equation is derived from the conservation of mass for the fluid inside the fracture, the Poiseuille law and an appropriate pressure law (see [15, Section 3] and [14] for further details). In [15], weak solutions are constructed by passing to the limit in a regularized problem. The necessary compactness estimates are obtained from appropriate energy estimates.

The equation under consideration

$$u_t + \partial_x(u^n \partial_x I(u)) = 0 \tag{2}$$

is a nonlocal degenerate parabolic equation of order $\alpha + 2$.

When $\alpha = 2$, this equation coincides with the thin film equation (TFE for short)

$$u_t + \partial_x(u^n \partial^3_{xxx} u) = 0. \tag{3}$$

This is a fourth order nonlinear degenerate parabolic equation originally studied by Bernis and Friedman [3]. This equation arises in many applications like spreading of a liquid film over a solid surface ($n = 3$) and Hele-Shaw flows ($n = 1$) (see [10–12,16,6,5,2]). TFE is derived also from a conservation of mass, the Poiseuille law (derived from a lubrication approximation of the Navier–Stokes equations for thin film viscous flows) and various pressure laws. The parameter $n \in (0,3]$ models various boundary conditions at the liquid–solid interface. The case $n > 3$ is mainly of mathematical interest [13]. In [3] weak solutions u are exhibited in a bounded interval under appropriate boundary conditions. In addition, they proved that u is nonnegative if u_0 is also so, and that the support of the solution $u(t,.)$ increases with t if u_0 is nonnegative and $n \geq 4$.

For $\alpha = 0$, the porous medium equation (PME for short) is recovered

$$u_t - \partial_x(u^n \partial_x u) = 0. \tag{4}$$

This is a nonlinear degenerate parabolic equation. The simple PME model describes the modeling of the motion of a gas flow through a porous medium [20]. In this case, the PME is derived from mass balance, Darcy's law which describes the dynamics of flows through porous media, and a state equation for the pressure [20]. PME also arises in heat transfer [18] and groundwater flow [19] and was originally proposed by Boussinesq. It took many years to prove that PME is well posed and the famous source type solutions were found by Zel'dovich, Kompanyeets and Barenblatt [20]. The questions of existence, uniqueness, stability, smoothness of solutions together with dynamical properties and asymptotic behavior are well represented in [20] where