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ScienceDirect
J. Differential Equations 259 (2015) 5813–5845
Journal of Differential Equations
www.elsevier.com/locate/jde

2D Grushin-type equations: Minimal time and null controllable data

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Received 12 November 2014; revised 19 May 2015

Available online 21 August 2015

Abstract

We study internal null controllability for degenerate parabolic equations of Grushin-type $G_\gamma = \partial^2_{xx} + |x|^{2\gamma}\partial^2_{yy}$ ($\gamma > 0$), in the rectangle $(x, y) \in \Omega = (-1, 1) \times (0, 1)$.

Previous works proved that null controllability holds for weak degeneracies (γ small), and fails for strong degeneracies (γ large). Moreover, in the transition regime and with strip shaped control domains, a positive minimal time is required.

In this paper, we work with controls acting on two strips, symmetric with respect to the degeneracy. We give the explicit value of the minimal time and we characterize some initial data that can be steered to zero in time T (when the system is not null controllable): their regularity depends on the control domain and the time T.

We also prove that, with a control that acts on one strip, touching the degeneracy line $\{x = 0\}$, then Grushin-type equations are null controllable in any time $T > 0$ and for any degeneracy $\gamma > 0$.

Our approach is based on a precise study of the observability property for the one-dimensional heat equations satisfied by the Fourier coefficients in variable y. This precise study is done, through a transmutation process, on the resulting one-dimensional wave equations, by lateral propagation of energy method.

MSC: 35K65; 93B05; 93B07

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http://dx.doi.org/10.1016/j.jde.2015.07.007
0022-0396/

Keywords: Grushin operator; Null controllability; Minimal time; Geometric control condition; Carleman estimates; Transmutation

1. Introduction

1.1. Main results

We consider Grushin-type equations

$$\begin{cases} \partial_t f - \partial_{xx}^2 f - |x|^{2\gamma}\partial_{yy}^2 f = u(t,x,y)\mathbf{1}_\omega(x,y), & (t,x,y)\in(0,T)\times\Omega, \\ f(t,x,y)=0, & (t,x,y)\in(0,T)\times\partial\Omega, \\ f(0,x,y)=f^0(x,y), & (x,y)\in\Omega, \end{cases} \tag{1.1}$$

where $\Omega := (-1,1)\times(0,1)$, $\gamma>0$ and $\mathbf{1}_\omega$ denotes the characteristic function of the subset ω. It is a degenerate parabolic equation, since the coefficient of $\partial_{yy}^2 f$ vanishes on the line $\{x=0\}$. System (1.1) is a linear control system in which the state is f and the control is the locally distributed source term u. We are interested in its null controllability, in the following sense.

Definition 1.1 *(Null controllability).* Let $T>0$ and $\omega\subset\Omega$. System (1.1) is null controllable from ω in time T if, for every $f^0\in L^2(\Omega,\mathbb{R})$, there exists $u\in L^2((0,T)\times\Omega,\mathbb{R})$ such that the associated solution of (1.1) satisfies $f(T,.,.)=0$.

System (1.1) is null controllable from ω if there exists $T>0$ such that system (1.1) is null controllable from ω in time T.

In [6], Beauchard, Cannarsa and Guglielmi proved the following result.

Theorem 1.1. *Let ω be an open subset of $(-1,1)\times(0,1)$ such that $\overline{\omega}\subset(0,1]\times[0,1]$.*

1. *If $\gamma\in(0,1)$, then system (1.1) is null controllable from ω in any time $T>0$.*
2. *If $\gamma=1$ and $\omega=(a,b)\times(0,1)$ where $0<a<b\leqslant 1$, then a positive minimal time is required for null controllability from ω; moreover*

$$T_{min} := \inf\{T>0;\ \textit{system (1.1) is null controllable from } \omega \textit{ in time } T\} \tag{1.2}$$

   satisfies $T_{min}\geqslant \frac{a^2}{2}$.
3. *If $\gamma>1$, then system (1.1) is not null controllable from ω.*

In particular, null controllability holds for weak degeneracies ($0<\gamma<1$), fails for strong degeneracies ($\gamma>1$) and, in the transition regime ($\gamma=1$), a positive minimal time is required.

The goal of the present article is to go further in this direction, and to give

- the explicit value of the minimal time T_{min},
- a characterization of initial conditions that can be steered to zero, when the system is not null controllable.