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# On hyperbolic equations and systems with non-regular time dependent coefficients

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**Abstract**

In this paper we study higher order weakly hyperbolic equations with time dependent non-regular coefficients. The non-regularity here means less than Hölder, namely bounded coefficients. As for second order equations in [14] we prove that such equations admit a 'very weak solution' adapted to the type of solutions that exist for regular coefficients. The main idea in the construction of a very weak solution is the regularisation of the coefficients via convolution with a mollifier and a qualitative analysis of the corresponding family of classical solutions depending on the regularising parameter. Classical solutions are recovered as limit of very weak solutions. Finally, by using a reduction to block Sylvester form we conclude that any first order hyperbolic system with non-regular coefficients is solvable in the very weak sense.


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## 1. Introduction

We want to study equations of the type

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$$D_t^m u - \sum_{j=1}^{m} \sum_{|\nu|=j} a_{\nu,j}(t) D_t^{m-j} D_x^\nu u - \sum_{j=1}^{m} \sum_{|\nu|<j} b_{\nu,j}(t) D_t^{m-j} D_x^\nu u = f(t,x),$$
$$t \in [0,T], x \in \mathbb{R}^n, \tag{1}$$

under initial conditions

$$D_t^k u(0,x) = g_k, \quad k = 0, \cdots, m-1. \tag{2}$$

We assume that the roots of the characteristic polynomial

$$\tau^m - \sum_{j=1}^{m} \sum_{|\nu|=j} a_{\nu,j}(t) \xi^\nu \tau^{m-j} = \Pi_{j=1,\dots,m}(\tau - \lambda_j(t,\xi))$$

are real and bounded in $t$ but not necessarily regular, for instance they might be discontinuous in $t$ as generated by discontinuous coefficients $a_{\nu,m}$. We assume that the coefficients of the lower order terms are compactly supported distributions with support contained in $[0,T]$, the right-hand side $f$ belongs to $\mathcal{E}'([0,T]) \otimes \mathcal{E}'(\mathbb{R}^n)$ and that the initial data belong to $\mathcal{E}'(\mathbb{R}^n)$.

Typical examples are the wave equation

$$\partial_t^2 u(t,x) - \sum_{i=1}^{n} a_i(t) \partial_{x_i}^2 u(t,x) = f(t,x),$$

where the coefficients $a_i$ are Heaviside functions or more in general equations of the type

$$D_t^2 u(t,x) - \sum_{i=1}^{n} b_i(t) D_t D_{x_i} u(t,x) - \sum_{i=1}^{n} a_i(t) D_{x_i}^2 u(t,x) = f(t,x), \tag{3}$$

where the coefficients are bounded real valued functions with $a_i$ positive for all $i = 1, \dots, n$ (see [14] for more details). Note that it is not restrictive to assume that the coefficients are compactly supported as in [14]. An immediate higher order example is given by the composition of a finite number of hyperbolic second order operators as in (3), i.e.,

$$\left(\Pi_{k=1}^m \left(D_t^2 - \sum_{i=1}^{n} b_{k,i}(t) D_t D_{x_i} - \sum_{i=1}^{n} a_{k,i}(t) D_{x_i}^2\right)\right) u(t,x)$$

plus lower order terms. Its characteristic polynomial

$$\Pi_{k=1}^m \left(\tau^2 - \sum_{i=1}^{n} b_{k,i}(t) \tau \xi_i - \sum_{i=1}^{n} a_{k,i}(t) \xi_i^2\right)$$

has $2m$ real roots.

Hyperbolic Cauchy problems with non-regular coefficients naturally appear in applied sciences as geophysics and seismology, to model delta-like sources and discontinuous or more irregular media. We refer the reader to [19] and [15] for a survey on this kind of applications.