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Analysis of non-isentropic compressible Euler equations with relaxation

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Abstract

This paper is contributed to the study of the one-dimensional non-isentropic compressible Euler equations with relaxation. It is shown that classical solutions do not exist globally-in-time under general conditions on initial data. Indeed, finite-time blowup occurs in a quantity related to the first moment. On the other hand, when the initial datum is sufficiently close to a constant equilibrium state, it is shown that the equations possess a unique global-in-time classical solution, and the solution converges to the equilibrium state in the long-time run. When the domain is finite, the convergence rate is shown to be exponential, due to boundary effects.

MSC: 35L50; 35L65

Keywords: Non-isentropic compressible Euler equations; Relaxation; Finite-time blowup; Global existence; Long-time behavior

1. Introduction

The phenomenon of relaxation occurs in many physical situations, such as kinetic theory, nonequilibrium gas dynamics, chromatography, elasticity with memory, flood flow with friction,

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traffic flow and magneto-hydrodynamics, etc., see [42] or [35]. From the point view of mathematics, the investigation of quantitative and qualitative behavior of solutions to relaxation systems is an important subject. A good survey in this direction is [35].

In this paper, we consider the one-dimensional non-isentropic compressible Euler equations with relaxation in Eulerian coordinates:

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2) + \partial_x P = \dfrac{\rho}{\tau}\big(f(\rho) - u\big), \\ \partial_t(\rho E) + \partial_x\left(u\left(\rho E + P\right)\right) = \dfrac{\rho u}{\tau}\big(f(\rho) - u\big) + \dfrac{\rho}{\tau}\big(g(\rho) - \theta\big), \end{cases} \tag{1.1}$$

$x \in \mathbb{R}$, $t > 0$, where ρ, u, P, E, θ denote the flow density, velocity, pressure, total energy and temperature, respectively, and $\tau > 0$ is the relaxation parameter. The constitutive relations are given by

$$P = \rho\theta, \quad E = \frac{1}{2}u^2 + \frac{\theta}{\gamma - 1}, \quad \theta = \rho^{\gamma-1}e^s,$$

where s is the entropy, $\gamma > 1$ is the adiabatic gas exponent. Here f and g are given smooth functions of ρ, whose dependence on ρ will be specified later. In this paper, we are interested in the global existence/finite-time singularity and long-time asymptotic behavior of the model for $\tau > 0$. Throughout this paper, we take $\tau = 1$.

This work is strongly motivated by the following improved traffic flow model proposed by Helbing [11]:

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t u + u\partial_x u + \dfrac{1}{\rho}\partial_x(\rho\theta) = \dfrac{1}{\tau}\big(u_e(\rho) - u\big), \\ \partial_t \theta + u\partial_x\theta + 2\theta\partial_x u = \dfrac{2}{\tau}\big(\theta_e(\rho) - \theta\big), \end{cases} \tag{1.2}$$

$x \in \mathbb{R}$, $t > 0$, which is an extension to the second order fluid-dynamic traffic models [1,7–9,12,21,24,38,42,47] by adding an equation for the vehicles' velocity variance. For smooth solutions, (1.2) can be written as a special case of (1.1) when $\gamma = 3$, $f(\rho) = u_e(\rho)$ and $g(\rho) = \theta_e(\rho)$. The unknown functions ρ, u, θ are interpreted as the traffic flow density, velocity and velocity variance, respectively. $u_e(\rho)$ and $\theta_e(\rho)$ are the equilibrium velocity and the equilibrium velocity variance, respectively. The system of three equations describes the conservation of number of vehicles on the road, dynamics of vehicle velocity or the acceleration behavior and dynamics of velocity variance. We point out that, the third order improved model is able to describe the observed increase of velocity variance immediately before a traffic jam develops, which cannot be described by the second order isentropic model (i.e., (1.2) with $s = \text{const.}$). Numerical simulations obtained in [11] showed consistency with major experimental observations.