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Asymptotic behavior of degenerate logistic equations ☆

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Abstract

We analyze the asymptotic behavior of positive solutions of parabolic equations with a class of degenerate logistic nonlinearities of the type $\lambda u - n(x)u^{\rho}$. An important characteristic of this work is that the region where the logistic term $n(\cdot)$ vanishes, that is $K_0 = \{x : n(x) = 0\}$, may be non-smooth. We analyze conditions on λ, ρ, $n(\cdot)$ and K_0 guaranteeing that the solution starting at a positive initial condition remains bounded or blows up as time goes to infinity. The asymptotic behavior may not be the same in different parts of K_0.

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1. Introduction

Let us consider the following reaction diffusion problem of logistic type

$$\begin{cases} u_t - \Delta u = \lambda u - n(x)u^\rho & \text{in } \Omega,\ t > 0 \\ u = 0 & \text{on } \partial\Omega,\ t > 0 \\ u(0) = u_0 \geq 0 \end{cases} \tag{1.1}$$

in a bounded domain $\Omega \subset I\!R^N$, $N \geq 1$, where $n(x) \geq 0$ in Ω is a bounded function not identically zero, $\rho > 1$, and $\lambda \in I\!R$. Observe that (1.1) is well posed for $0 \leq u_0 \in L^1(\Omega)$ and the solution, which will be denoted $u(t; u_0)$, becomes classical and positive for $t > 0$, see Section 2.

As for the asymptotic behavior of nonnegative solutions, note that (1.1) degenerates to a linear equation on the set in which $n(x)$ vanishes, denoted by K_0. This set plays a crucial role in the dynamical properties and the asymptotic behavior of solutions of (1.1). In fact, one expects the solution to grow in this set, at least for large values of λ. On the other hand, in the complementary set of K_0 the nonlinear reaction term acts, and one expects the solutions to remain bounded in this set. Therefore, to determine the long term behavior of solutions, there should be a balance between the size of K_0 and the strength of the nonlinear term (measured in terms of ρ and the way that $n(x)$ vanishes near K_0) which we try to unveil here.

This type of question was studied before but always under the assumption that the function $n(x)$ is somehow smooth and the set $\{x \in \Omega : n(x) > 0\}$ is an open subset of Ω with regular boundary. Without pretending being exhaustive, we would like to mention some relevant contributions in this direction. For instance, Ouyang [25] studied the elliptic problem associated to (1.1). Then Fraile et al. [12] considered the elliptic but also the parabolic problem for the first time in the literature. Later Du and Huang [10] improved some of the technical conditions of the previous paper and proved results related to those of [16]. Since then, many papers have been devoted to this subject, many of them related to understanding the behavior of the solutions of some singular elliptic boundary value problem associated to (1.1); see e.g. [13–15]. A rather complete survey on this subject is [22]. All these papers contain references to other related results. Finally, see [26] for similar results in unbounded domains.

As a difference with respect to the above mentioned contributions, in the present paper we make no regularity assumption at all on $n(x)$ nor on K_0, other than $n(x)$ being bounded and $K_0 \subset \Omega$ compact.

One particular and relevant example that may help us to grasp the scope of our results is the following. In dimension $N \geq 2$, consider the set K_0 is decomposed as $K_0 = K_1 \cup K_2$, where $K_1 = \overline{B}$ is the closure of a bounded, smooth, connected domain B (think for instance B is a ball) and K_2 is a segment (see Fig. 1 and ignore the dotted box appearing in the figure for a moment). If we denote by $\lambda_1(D)$ the first eigenvalue of the Laplacian operator with Dirichlet boundary conditions in a generic domain $D \subset I\!R^N$, we get the following picture. For $\lambda < \lambda_1(B)$ all solutions of (1.1) converge to a unique smooth globally asymptotically stable equilibria of (1.1). This equilibria is positive in Ω for $\lambda_1(\Omega) < \lambda < \lambda_1(B)$ and identically zero otherwise. On the other hand for $\lambda \geq \lambda_1(B)$ then all solutions of (1.1) become unbounded in a neighborhood of K_0 and bounded in the rest of Ω.

If K_0 is disconnected, that is $K_1 \cap K_2 = \emptyset$ then for all $\lambda \in I\!R$ all solutions remain bounded in K_2. On the other hand, if $K_1 \cap K_2$ is a point, as depicted in Fig. 1 and we assume that

$$n(x) \geq \operatorname{dist}(x, K_0)^\gamma \qquad \text{for some} \quad \gamma > 0$$