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Existence of a solution to an equation arising from the theory of Mean Field Games

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Abstract

We construct a small time strong solution to a nonlocal Hamilton–Jacobi equation (1.1) introduced in [48], the so-called master equation, originating from the theory of Mean Field Games. We discover a link between metric viscosity solutions to local Hamilton–Jacobi equations studied in [2,19,20] and solutions to (1.1). As a consequence we recover the existence of solutions to the First Order Mean Field Games equations (1.2), first proved in [48], and make a more rigorous connection between the master equation (1.1) and the Mean Field Games equations (1.2).

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Contents

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1. Introduction

The theory of Mean Field Games (MFG) analyzes differential games with a large number of players, each player having a very little influence on the overall system. This theory, which encompasses games with a continuum of players, was developed by Lasry–Lions [44–47]. Similar ideas were independently introduced at the same time and studied in the engineering literature by Huang–Caines–Malhamé [36–38,40]. Games with a continuum of players or traders, first appeared in economics, starting with the seminal work of Aumann [5]. Later a theory of non-atomic games was presented in a book by Aumann–Shapley [6]. In this pioneering work, Aumann–Shapley proposed a profound mathematical theory for economics, the potential of which has not yet been fully exploited. The term "Mean Field Games" was introduced by analogy with the mean field models in mathematical physics where the behaviors of many identical particles are analyzed. We refer the readers to [9,12,22,29,32] for several excellent surveys on the theory of MFG and its extensions. In particular, the notes [12] from the lectures of P.-L. Lions [48] have been a great contribution to the field, and have clarified the current state of the theory of MFG. This was the starting point of our study.

The theory of MFG has attracted significant attention. In the past five years alone, a large number of manuscripts have been devoted to it, revealing its importance, impact, and possible applications (see e.g. [7,8,15,23–28,30,31,33–35,39,41–43,49–52]). In light of the publications [44–47], we restrict our study to the simplest framework of games: those with identical players. Our effort will be devoted mainly to the study of the master equation of MFG (1.1); only a small part of the manuscript deals with the MFG equations (1.2) which were studied in [46,48,12]. Our main result establishes the short time existence of a regular solution to (1.1).