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On cell problems for Hamilton–Jacobi equations with non-coercive Hamiltonians and their application to homogenization problems

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Abstract

We study a cell problem arising in homogenization for a Hamilton–Jacobi equation whose Hamiltonian is not coercive. We introduce a generalized notion of effective Hamiltonians by approximating the equation and characterize the solvability of the cell problem in terms of the generalized effective Hamiltonian. Under some sufficient conditions, the result is applied to the associated homogenization problem. We also show that homogenization for non-coercive equations fails in general.

MSC: 35F21; 35D40; 35B27; 49L25; 74N05

Keywords: Cell problem; Homogenization; Hamilton–Jacobi equation; Non-coercive Hamiltonian; Viscosity solution; Generalized effective Hamiltonian

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1. Introduction

We consider a Hamilton–Jacobi equation of the form

$$H(x, Du(x) + P) = a \quad \text{in } \mathbf{T}^N \tag{CP}$$

and study a problem to find, for a given $P \in \mathbf{R}^N$, a pair of a function $u: \mathbf{T}^N \to \mathbf{R}$ and a constant $a \in \mathbf{R}$ such that u is a Lipschitz continuous viscosity solution of (CP). Here, $\mathbf{T}^N := \mathbf{R}^N/\mathbf{Z}^N$ and a function u on $\mathbf{T}^N$ is regarded as a function defined on $\mathbf{R}^N$ with $\mathbf{Z}^N$-periodicity, i.e., $u(x + z) = u(x)$ for all $x \in \mathbf{R}^N$ and $z \in \mathbf{Z}^N$. Moreover, Du denotes the gradient, i.e., $Du = (\partial u/\partial x_1, \dots, \partial u/\partial x_N)$. This kind of problem is called a *cell problem* in the theory of homogenization. The constant a satisfying (CP) is called a *critical value* if it is uniquely determined.

As a typical example in this paper, we consider the Hamiltonian $H : \mathbf{T}^N \times \mathbf{R}^N \to \mathbf{R}$ in (CP) given by

$$H(x, p) = \sigma(x)m(|p|), \tag{1.1}$$

where σ and m satisfy

(H1) $\sigma: \mathbf{T}^N \to (0, \infty)$ is a continuous function,
(H2) $m: [0, \infty) \to (0, 1)$ is a Lipschitz continuous function,
(H3) m is strictly increasing and $m(r) \to 1$ as $r \to \infty$.

Due to the boundedness of m, our cell problem does not necessarily admit a solution (u, a), and the solvability depends on $P \in \mathbf{R}^N$. One of goals in this paper is to characterize the set of $P \in \mathbf{R}^N$ such that the cell problem admits a solution. The other goal is to apply the result to the associated homogenization problem.

A result for existence of a solution of cell problems for Hamilton–Jacobi equations was first established by Lions, Papanicolaou and Varadhan [20] under the assumption that the Hamiltonian is *coercive*, i.e.,

$$\lim_{r\to\infty} \inf\{H(x, p) \mid x \in \mathbf{T}^N,\ p \in \mathbf{R}^N,\ |p| \geq r\} = +\infty. \tag{1.2}$$

Their method begins with considering the following approximate equation with a parameter $\delta > 0$:

$$\delta u_\delta(x) + H(x, Du_\delta(x) + P) = 0 \quad \text{in } \mathbf{T}^N. \tag{1.3}$$

By a standard argument of viscosity solutions, it turns out that there exists a unique solution u_δ and that a family of functions $\{\delta u_\delta\}_{\delta>0}$ is uniformly bounded. Thus, (formally) $\{Du_\delta\}_{\delta>0}$ is uniformly bounded thanks to the coercivity. Therefore, by taking a subsequence if necessary, δu_δ and $u_\delta - \min u_\delta$ uniformly converge to a constant $-a$ and a function u as $\delta \to 0$, respectively. A stability argument of viscosity solutions shows that u and a solve (CP). For more details, see [20] and [13]. We point out that the paper [13] also studies second order uniformly elliptic equations by using a similar argument.

Unfortunately, our Hamiltonian (1.1) is not coercive because of the boundedness of the function m. When a Hamiltonian is not coercive, the method of [20] becomes very delicate.