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Nonlocal systems of balance laws in several space dimensions with applications to laser technology

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Abstract

For a class of systems of nonlinear and nonlocal balance laws in several space dimensions, we prove the local in time existence of solutions and their continuous dependence on the initial datum. The choice of this class is motivated by a new model devoted to the description of a metal plate being cut by a laser beam. Using realistic parameters, solutions to this model obtained through numerical integrations meet qualitative properties of real cuts. Moreover, the class of equations considered comprises a model describing the dynamics of solid particles along a conveyor belt.

MSC: 35L65

Keywords: Nonlocal balance laws; Laser cutting; Conveyor belt dynamics

1. Introduction

We are concerned with a system of n balance laws in several space dimensions of the type

$$\begin{cases} \partial_t u_i + \mathrm{div}_x\, \varphi_i(t,x,u_i,\vartheta * u) = \Phi_i(t,x,u_i,\vartheta * u) \\ u_i(0,x) = \bar{u}_i(x) \end{cases} \qquad i = 1,\dots,n\,. \tag{1.1}$$

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Here, $t \in [0, +\infty[$ is time, $x \in \mathbb{R}^N$ is the space coordinate and $u \equiv (u_1, \dots, u_n)$, with $u_i = u_i(t, x)$, is the unknown. The function ϑ is a smooth function defined in $\mathbb{R}^N$ attaining as values $m \times n$ matrices, so that

$$\vartheta \in \mathbf{C}^{\mathbf{2}}_{\mathbf{c}}(\mathbb{R}^N; \mathbb{R}^{m\times n}), \qquad (\vartheta * u(t))(x) = \int_{\mathbb{R}^N} \vartheta(x-\xi)\, u(t,\xi) \mathrm{d}\xi, \qquad (\vartheta * u(t))(x) \in \mathbb{R}^m.$$

The flow $\varphi \equiv (\varphi_1, \dots, \varphi_n)$, with $\varphi_i(t, x, u_i, A) \in \mathbb{R}^N$, and the source $\Phi \equiv (\Phi_1, \dots, \Phi_n)$, with $\Phi_i(t, x, u_i, A) \in \mathbb{R}$, have the peculiar property that the equations are coupled only through the nonlocal convolution term $\vartheta * u$.

The driving example for our considering the class (1.1) is a new model for the cutting of metal plates by means of a laser beam, presented in Section 3. A sort of *pattern formation* phenomenon, typical of various nonlocal equations [7], provides a first preliminary description of the formation of the well known *ripples* whose insurgence deeply affects the quality of the cuts. In fact, two types of lasers are mainly used in the cutting of metals: CO_2 lasers and fiber lasers. The former ones are more powerful and more precise, but also more expensive. Recent technological improvements are apparently going to allow also to the cheaper devices of the latter type to cut thick plates, nowadays treated typically with CO_2 lasers. Unfortunately, a typical drawback of fiber lasers is that along the cut ripples are generated, see [26,28,33]. The modeling of these ripples often relies on the introduction of *imperfections* in the metal or of *inaccuracies* in the laser management, see also [25,27]. Here, using realistic numeric parameters and in spite of the rather simplified physics involved, in Section 3 we present a first preliminary model capable of describing the formation of a geometry similar to the ripples observed in industrial cuts. We remark that in the present construction neither the initial data nor the parameters in the equations contain any oscillating term.

Furthermore, in Section 4, we slightly extend the model introduced in [12] to describe the dynamics of bolts along a conveyor belt. The resulting equations fit in the present framework and is proved to be well posed.

Besides, we also note that several crowd dynamics models considered in the literature fit into (1.1), e.g. [7,9,11,18].

The particular structure of (1.1) allows to prove its well posedness. Indeed, for small times, system (1.1) admits a unique solution $u = u(t, x)$. Moreover, u is proved to be a continuous function of time with respect to the $\mathbf{L^1}$ topology and an $\mathbf{L^1}$-Lipschitz continuous function of the initial datum $\bar{u}$. In all this, the particular coupling among the equations in (1.1) plays a key role. At present, the well posedness of general systems of balance laws in several space dimensions is a formidable open problem. In the present work, the functional setting is provided by $\mathbf{L^1} \cap \mathbf{L^\infty} \cap \mathbf{BV}$, as usual in the framework of nonlocal conservation laws. The existence result is obtained through a careful use of the general estimates [10,21]. They provide the necessary analytic tool to apply Banach Contraction Theorem.

A preliminary result related to Theorem 2.2 below is presented for instance in [1], see also [3]. There, the existence of solution to (1.1) in the case $\Phi \equiv 0$ is obtained proving the convergence (up to a subsequence) of a Lax–Friedrichs type approximate solutions. Note however that differently from the present situation, in the case considered in [1], positive initial data yield positive solutions so that the $\mathbf{L^1}$ norm is conserved.

We remark that most of the results related to nonlocal balance laws are currently devoted to conservation laws, i.e., to equations that lack any source term. Here, we allow for the presence