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First-order aggregation models and zero inertia limits ☆

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Abstract

We consider a first-order aggregation model in both discrete and continuum formulations and show rigorously how it can be obtained as zero inertia limits of second-order models. In the continuum case the procedure consists in a macroscopic limit, enabling the passage from a kinetic model for aggregation to an evolution equation for the macroscopic density. We work within the general space of measure solutions and use mass transportation ideas and the characteristic method as essential tools in the analysis.

Keywords: Aggregation models; Kinetic equations; Macroscopic limit; Measure solutions; Mass transportation; Particle methods

1. Introduction

The focus of the present paper is a certain mathematical model for emerging self-collective behaviour in biological (and other) aggregations. There has been a surge of activity in this area of research during the past decade, and in fact the goals have extended well beyond biology. For biological applications, the primary motivation has been to understand and model the mechanisms behind the formation of the various spectacular groups observed in nature (fish schools, bird flocks, insect swarms) [11]. In terms of expansion of this research into collateral areas, we men-

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tion studies on robotics and space missions [27], opinion formation [33], traffic and pedestrian flow [23] and social networks [26].

Aggregation models can be classified in two main classes: i) individual/particle-based, where the movements of all individuals in the group are being tracked, and ii) partial differential equations (PDE) models, formulated as evolution equations for the population density field. We refer to [14] for a recent review of models for aggregation behaviour, where the various microscopic/macroscopic descriptions of collective motion are discussed and connected. In the present work we deal with a model that has both a discrete/ODE and a continuum/PDE formulation.

The continuum aggregation model considered in this article is given by the following evolution equation for the population density $\rho(t,x)$ in $\mathbb{R}^d$:

$$\rho_t + \nabla \cdot (\rho u) = 0, \tag{1.1a}$$

$$u = -\nabla K * \rho, \tag{1.1b}$$

where K represents an interaction potential and $*$ denotes convolution. The potential K typically incorporates social interactions such as short-range repulsion and long-range attraction. We consider K to be radial, meaning that the inter-individual interactions are assumed to be isotropic.

Equation (1.1) appears in various contexts related to mathematical models for biological aggregations; we refer to [32,36] and the references therein for an extensive background and review of the literature on this topic. It also arises in a number of other applications such as material science and granular media [37], self-assembly of nanoparticles [24] and molecular dynamics simulations of matter [22]. The model has become widely popular and there has been intensive research on it during recent years.

The particular appeal of model (1.1) has lain in part in its simple form, which allowed rapid progress in terms of both numerics and analysis. Numerical simulations demonstrated a wide variety of self-collective or “swarm” behaviours captured by model (1.1), resulting in aggregations on disks, annuli, rings, soccer balls, etc. [28,39,40]. Analysis-oriented studies addressed the well-posedness of the initial-value problem for (1.1) [9,10,30,5,13,6], as well as the long time behaviour of its solutions [10,31,17,4,20,19]. Also, there has been increasing interest lately on the analysis of (1.1) by variational methods [3,2,15].

Equation (1.1) is frequently regarded as the continuum approximation, when the number of particles increases to infinity, of the following individual-based model. Consider N particles in $\mathbb{R}^d$ whose positions x_i ($i = 1, \ldots, N$) evolve according to the ODE system

$$\frac{dx_i}{dt} = v_i, \tag{1.2a}$$

$$v_i = -\frac{1}{N}\sum_{j\neq i} \nabla_{x_i} K(x_i - x_j), \tag{1.2b}$$

where K denotes the same interaction potential as in (1.1).

Model (1.2) was justified and formally derived in [8], starting from the following second-order model in Newton’s law form ($i = 1, \ldots, N$):

$$\epsilon \frac{d^2 x_i}{dt^2} + \frac{dx_i}{dt} = F_i, \qquad \text{with} \quad F_i = -\frac{1}{N}\sum_{j\neq i} \nabla_{x_i} K(x_i - x_j), \tag{1.3}$$